Comparison Sorts III continued

CS 311 Data Structures and Algorithms
Lecture Slides
Monday, October 7, 2024

Glenn G. Chappell
Department of Computer Science
University of Alaska Fairbanks
ggchappell@alaska.edu

Some material contributed by Chris Hartman

Unit Overview
Algorithmic Efficiency & Sorting

Topics

- ✓ Analysis of Algorithms
- ✓ Introduction to Sorting
- ✓ Comparison Sorts I
- ✓ Asymptotic Notation
- ✓ Divide and Conquer
- ✓ Comparison Sorts II
- ✓ The Limits of Sorting
- (part) Comparison Sorts III
- Non-Comparison Sorts
- Sorting in the C++ STL

Review

Review
Analysis of Algorithms

Cannot read all of input

Probably not scalable

Using Big-*O*	In Words
$O(1)$	Constant time
$O(\log n)$	Logarithmic time
$O(n)$	Linear time
$O(n \log n)$	Log-linear time
$O(n^2)$	Quadratic time
$O(c^n)$, for some $c > 1$	Exponential time

Faster

Slower

Useful Rules

- **Rule of Thumb.** For nested "real" loops, order is $O(n^t)$, where t is the number of nested loops.
- **Addition Rule.** $O(f(n)) + O(g(n))$ is either $O(f(n))$ or $O(g(n))$, *whichever is larger*. And similarly for Θ. This works when adding up any *fixed*, *finite* number of terms.

Introduction to Sorting — Overview of Algorithms

Sorting Algorithms Covered

- Quadratic-Time [$O(n^2)$] Comparison Sorts
  - ✓ Bubble Sort
  - ✓ Insertion Sort
  - (part) Quicksort
- Log-Linear-Time [$O(n \log n)$] Comparison Sorts
  - ✓ Merge Sort
  - Heap Sort (mostly later in semester)
  - Introsort
- Special Purpose—Not Comparison Sorts
  - Pigeonhole Sort
  - Radix Sort

Review
Asymptotic Notation

$g(n)$ is:

- $O(f(n))$ if $g(n) \leq k \times f(n)$ …
- $\Omega(f(n))$ if $g(n) \geq k \times f(n)$ …
- $\Theta(f(n))$ if both are true—possibly with different values of k.

Θ is very useful!

Ω not as much.

	1	n	$n \log n$	n^2	$5n^2$	$n^2 \log n$	n^3	n^4
$O(n^2)$	**YES**	**YES**	**YES**	**YES**	**YES**	no	no	no
$\Omega(n^2)$	no	no	no	**YES**	**YES**	**YES**	**YES**	**YES**
$\Theta(n^2)$	no	no	no	**YES**	**YES**	no	no	no

In an algorithmic context, $g(n)$ might be:

- The maximum number of basic operations performed by the algorithm when given input of size n.
- The maximum amount of additional space required.

In-place means using $O(1)$ additional space.

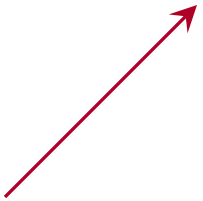
Review
Divide and Conquer [1/2]

A Divide/Decrease and Conquer algorithm needs analysis.

- It splits its input into b **nearly equal-sized** parts.
- It makes a recursive calls, each taking one part.
- It does other work requiring $f(n)$ operations.

To Analyze

- Find b, a, d so that $f(n)$ is $\Theta(n^d)$—or $O(n^d)$.
- Compare a and b^d.
- Apply the appropriate case of the Master Theorem.



The **Master Theorem**

Suppose $T(n) = a\,T(n/b) + f(n)$; $a \geq 1$, $b > 1$, $f(n)$ is $\Theta(n^d)$.

- "n/b" can be a nearby integer.

Then:

- Case 1. If $a < b^d$, then $T(n)$ is $\Theta(n^d)$.
- Case 2. If $a = b^d$, then $T(n)$ is $\Theta(n^d \log n)$.
- Case 3. If $a > b^d$, then $T(n)$ is $\Theta(n^k)$, where $k = \log_b a$.

We may also replace each "Θ" above with "O".

Divide and Conquer [2/2]

Try It!

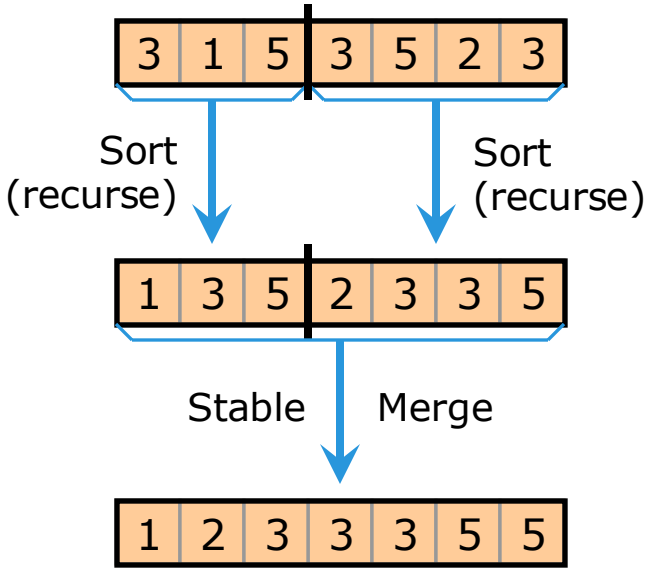
Algorithm *B* is given a list as input. It uses a Decrease and Conquer strategy. It splits its input in half (or nearly so), and makes a recursive call on one of the parts. It also does other work requiring linear time.

Use the Master Theorem to determine the order of Algorithm *B*.

See In-Class Worksheet 2: The Master Theorem.

Comparison Sorts II — Merge Sort

Merge Sort: recursively sort top & bottom halves of list, **merge**.



Analysis

- Efficiency: Θ(*n* log *n*). Avg same. ☺
- Requirements on Data: Works for Linked Lists, etc. ☺
- Space Efficiency: Θ(log *n*) space for recursion. Iterative version is in-place for Linked List. Θ(*n*) space for array. 😐/☺/☹
- Stable: Yes. ☺
- Performance on Nearly Sorted Data: Not better or worse. 😐

See `merge_sort.cpp`.

Notes

- Practical & often used.
- Fastest known for (1) stable sort, (2) sorting a Linked List.

The Limits of Sorting

The worst-case number of comparisons performed by a general-purpose comparison sort must be $\Omega(n \log n)$.

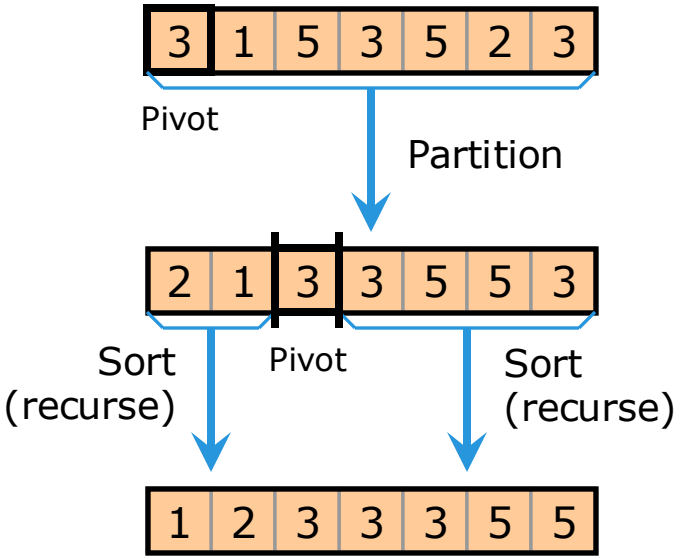
Reasoning:

- We are given a list of n items to be sorted.
- There are $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$ orderings of n items.
- Start with all $n!$ orderings. Do comparisons, throwing out orderings that do not match what we know, until just one ordering is left.
- With each comparison, we cannot guarantee that more than half of the orderings will be thrown out.
- How many times must we cut $n!$ in half, to get 1? Answer: $\log_2(n!)$, which is $\Theta(n \log n)$. (Use **Stirling's Approximation**.)
- So, for a general-purpose comparison sort, the worst-case number of comparisons must be *at least* that big. Thus: $\Omega(n \log n)$.

Comparison Sorts III — Quicksort [1/2]

Quicksort: choose **pivot**, **partition**, recursively sort sublists.

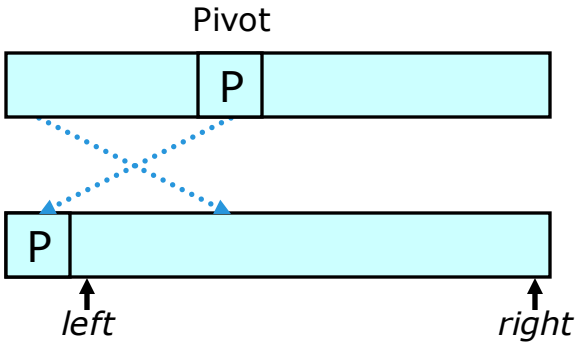
For the moment, we choose the first item in the list as our pivot.



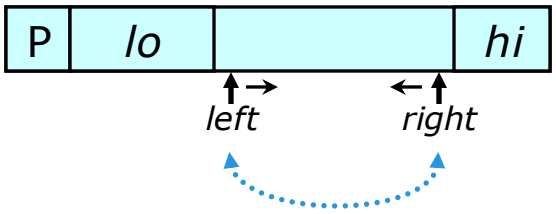
See `quicksort1.cpp`.

Hoare's Partition Algorithm

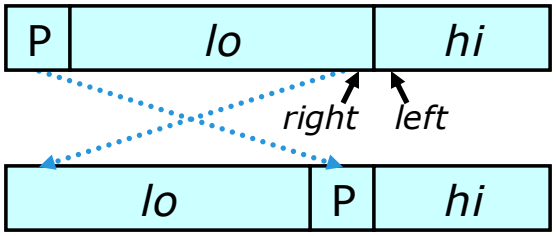
- First, get the pivot out of the way: swap it with the first list item.
- Set iterator *left* to point to the first item past the pivot. Set iterator *right* to point to the last list item.



- Move iterator *left* up, leaving only low items below it. Move iterator *right* down, leaving only high items above it.
- If both iterators get stuck—*left* points to a high item and *right* points to a low item—then swap the items and continue.



- Eventually *left* & *right* cross each other.
- Finish by swapping the pivot with the last low item.



Comparison Sorts III

continued

Better Quicksort — Problem

Quicksort has a serious problem.

- Try applying the Master Theorem. It does not work, because Quicksort may not split its input into nearly equal-sized parts.
- The pivot *might* be chosen very poorly. In such cases, Quicksort has linear recursion depth and does linear-time work at each step.
- Result: Quicksort is $\Theta(n^2)$. ☹
- And the worst case happens when the list is *already sorted*!

However, Quicksort's average-case time is very fast.

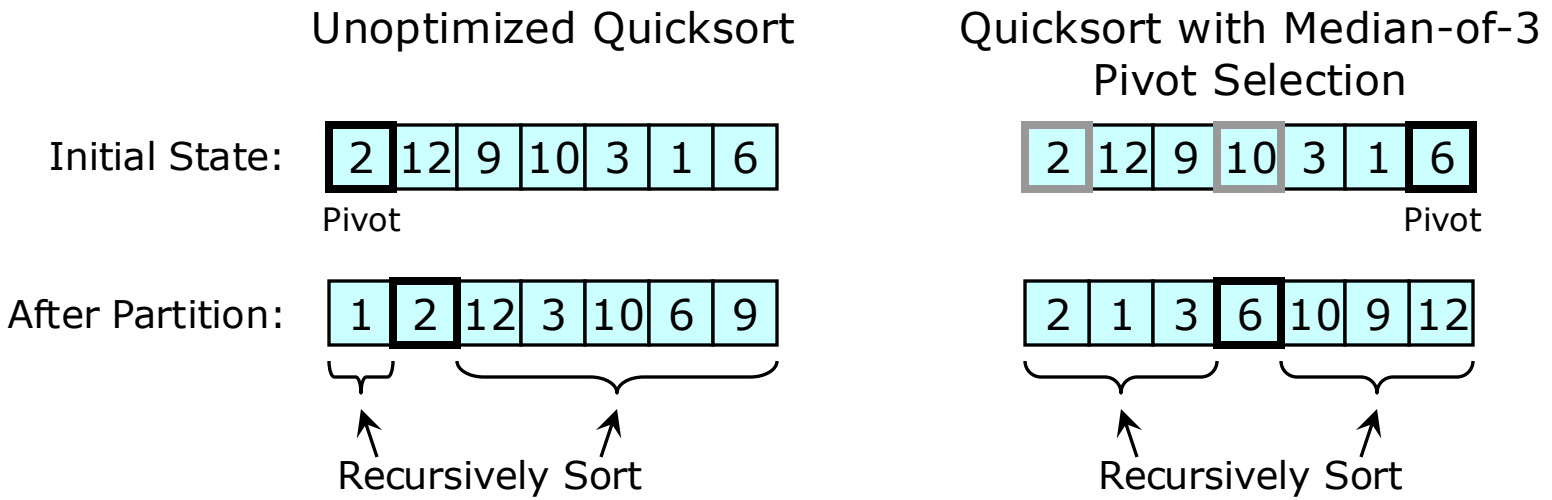
Quicksort is *usually* very fast, so people want to use it.

In the decades following Quicksort's introduction in 1961, many people published suggested improvements. We will look at three of the most successful.

Better Quicksort — Optimization 1: Improved Pivot Selection [1/2]

Choose the pivot using **Median-of-3**.

- Look at 3 items in the list: first, middle, last.
- Let the pivot be the one that is between the other two (by <).



This gives good performance on *most* nearly sorted data—as do other similar pivot-selection schemes.

But Quicksort with Median-of-3 (or similar) is slow for *other* data. So: still $\Theta(n^2)$.

You may wish to look into "Median-of-3 killer sequences".

Better Quicksort — Optimization 1: Improved Pivot Selection [2/2]

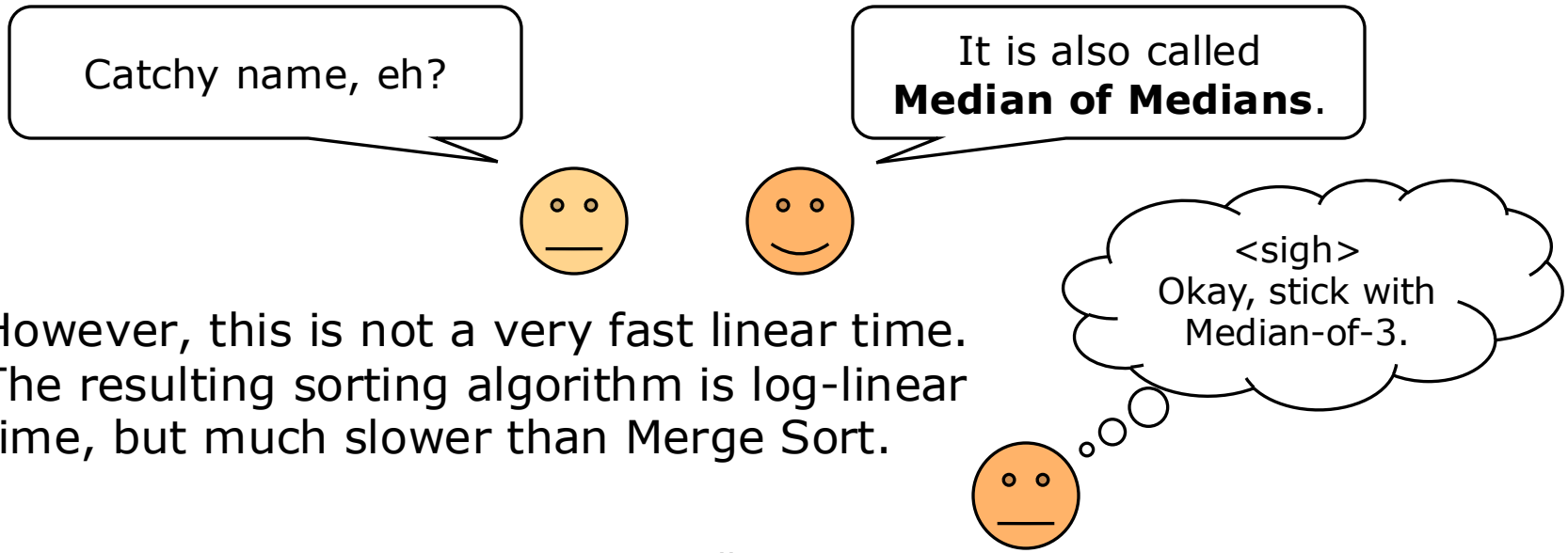
Ideally, our pivot is the *median* of the list.

Median: value that goes in the middle, when the list is sorted.

- If it were, then Partition would create lists of (nearly) equal size, and we could apply the Master Theorem, which would tell us:
- If we do $O(n)$ extra work at each step, then we get an $O(n \log n)$ algorithm (same computation as for Merge Sort).

Can we find the median of a list in linear time?

- Yes! Use **BFPRT** (the **B**lum-**F**loyd-**P**ratt-**R**ivest-**T**arjan Algorithm).



- However, this is not a very fast linear time. The resulting sorting algorithm is log-linear time, but much slower than Merge Sort.

Better Quicksort — Optimization 2: Tail-Recursion Elimination

How much additional space does Quicksort use?

- Partition is in-place and Quicksort uses few local variables.
- However, Quicksort is recursive.
- Quicksort's additional space usage is thus proportional to its recursion depth …
- … which is linear. Worst-case additional space used: $\Theta(n)$. ☹

We can significantly improve this:

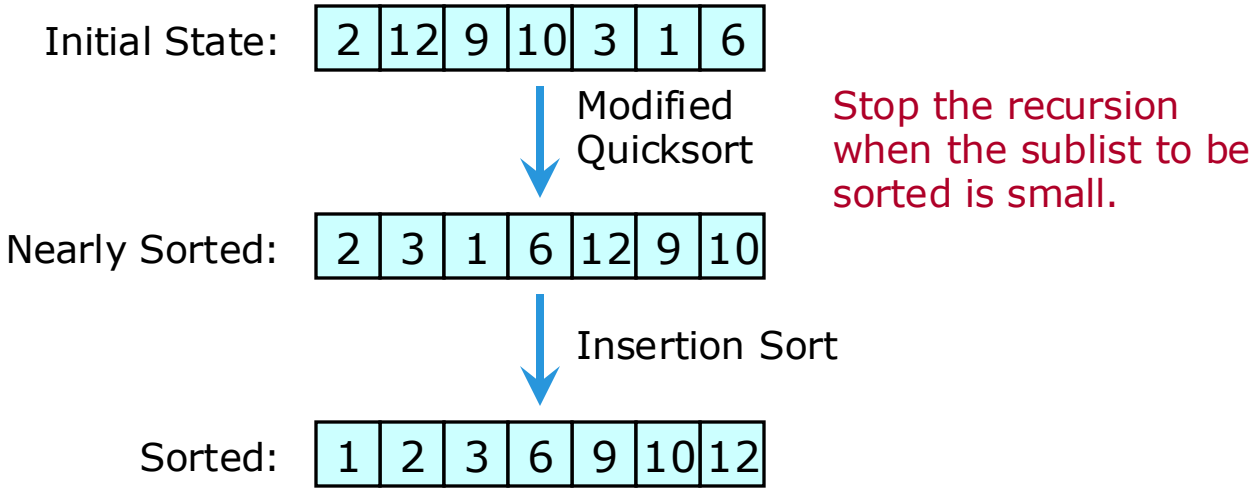
- Do the *larger* of the two recursive calls last.
- Do tail-recursion elimination on this final recursive call.
- Result: Recursion depth & additional space usage: $\Theta(\log n)$. 😐
- And this additional space need not hold data items. (Why is this kinda good?)

Better Quicksort — Optimization 3: Finishing with Insertion Sort

A *possible* speed-up: finish with Insertion Sort

- Quicksort, but without quite going to the bottom of the recursion. We end up with a *nearly sorted* list.
- Finish sorting this list using one call to Insertion Sort.
- Apparently this is generally faster*, but it is still $\Theta(n^2)$.

This is *not* the same as using Insertion Sort for small lists.



*I have read that this tends to adversely affect the number of cache hits.

Better Quicksort — CODE

TO DO

- Rewrite our Quicksort to include the optimizations discussed:
  - Median-of-3 pivot selection.
  - Tail-recursion elimination on the larger recursive call.
  - Recursive calls to sort small lists do nothing. End with Insertion Sort of entire list.

Done. See `quicksort2.cpp`.

Better Quicksort — What is Needed?

We want an algorithm that:

- Is as fast as Quicksort on average.
- Has good [$\Theta(n \log n)$] worst-case performance.

But for over three decades no one found one.

Some said (and some still say), "Quicksort's bad behavior is very rare; we can ignore it."

I suggest that this is not a good way to think.

- Sometimes poor worst-case behavior is okay; sometimes it is not.
- Know what is important in your situation.
- Remember that malicious users exist, particularly on the Web.

These are *general* principles. They apply to many issues, not just those involving Quicksort.

In 1997, a solution to Quicksort's big problem was finally published. We will discuss this. But first, we analyze Quicksort.

Better Quicksort — Analysis of Quicksort

This is Quicksort's advantage—its only advantage.

Efficiency ☹

- Quicksort is $\Theta(n^2)$.
- Quicksort has a very good $\Theta(n \log n)$ average-case time. ☺☺

Requirements on Data ☹

- Non-trivial pivot-selection algorithms (Median-of-3 and similar) are only efficient for random-access data.

Space Usage 😐

- Quicksort uses space for recursion.
  - Additional space: $\Theta(\log n)$, if clever tail-recursion elimination is done.
  - Even if *all* recursion is eliminated, $O(\log n)$ additional space is still used.
  - This additional space need not hold any data items.

Stability ☹

- Efficient versions of Quicksort are not stable.

Performance on Nearly Sorted Data 😐

- An unoptimized Quicksort is *slow* on nearly sorted data: $\Theta(n^2)$.
- Quicksort + Median-of-3 is $\Theta(n \log n)$ on *most* nearly sorted data.

Introsort — Introspection

In 1997, algorithms researcher David Musser introduced a simple algorithm-design idea.

- For some problems, there are known algorithms with very good average-case performance and very poor worst-case performance.
- Quicksort is the best known of these, but there are others.
- Musser's idea is that, when such an algorithm runs, it should keep track of its performance. If it is not doing well, then it can switch to a different algorithm that has a better worst-case.
- Musser called this technique **introspection**, since the algorithm is examining itself.

The most important application of introspection is to sorting. It allows us to eliminate the awful worst-case behavior of Quicksort.

Introsort — Heap Sort Preview

Here is a preview of a sort we will cover later in the semester.

We will study a data structure called a *Binary Heap*, which allows for fast find and removal of the item with the greatest key.

This leads to a comparison sort called **Heap Sort**. Procedure:

- Create a Binary Heap containing the dataset to be sorted.
- Repeatedly remove the item with the greatest key. Store these items in a list in reverse order: greatest at the end, etc.
- When complete, the list is a sorted version of the original dataset.

We study Heap Sort in detail later in the semester. For now:

- Heap Sort is log-linear time.
- Heap Sort is in-place.
- Heap Sort requires random-access data.

And Heap Sort forms part of a fast Quicksort variant called *Introsort*.

Introsort — Description

Quicksort's problem is due to its recursion depth. Quicksort is slow only *when the recursion gets too deep*.

Apply introspection:

- Do optimized Quicksort, but keep track of the recursion depth.
- If the depth exceeds some threshold—Musser suggested $2 \log_2 n$—then switch to Heap Sort for the current sublist being sorted.

The resulting algorithm is called **Introsort** [**intro**spective **sort**].

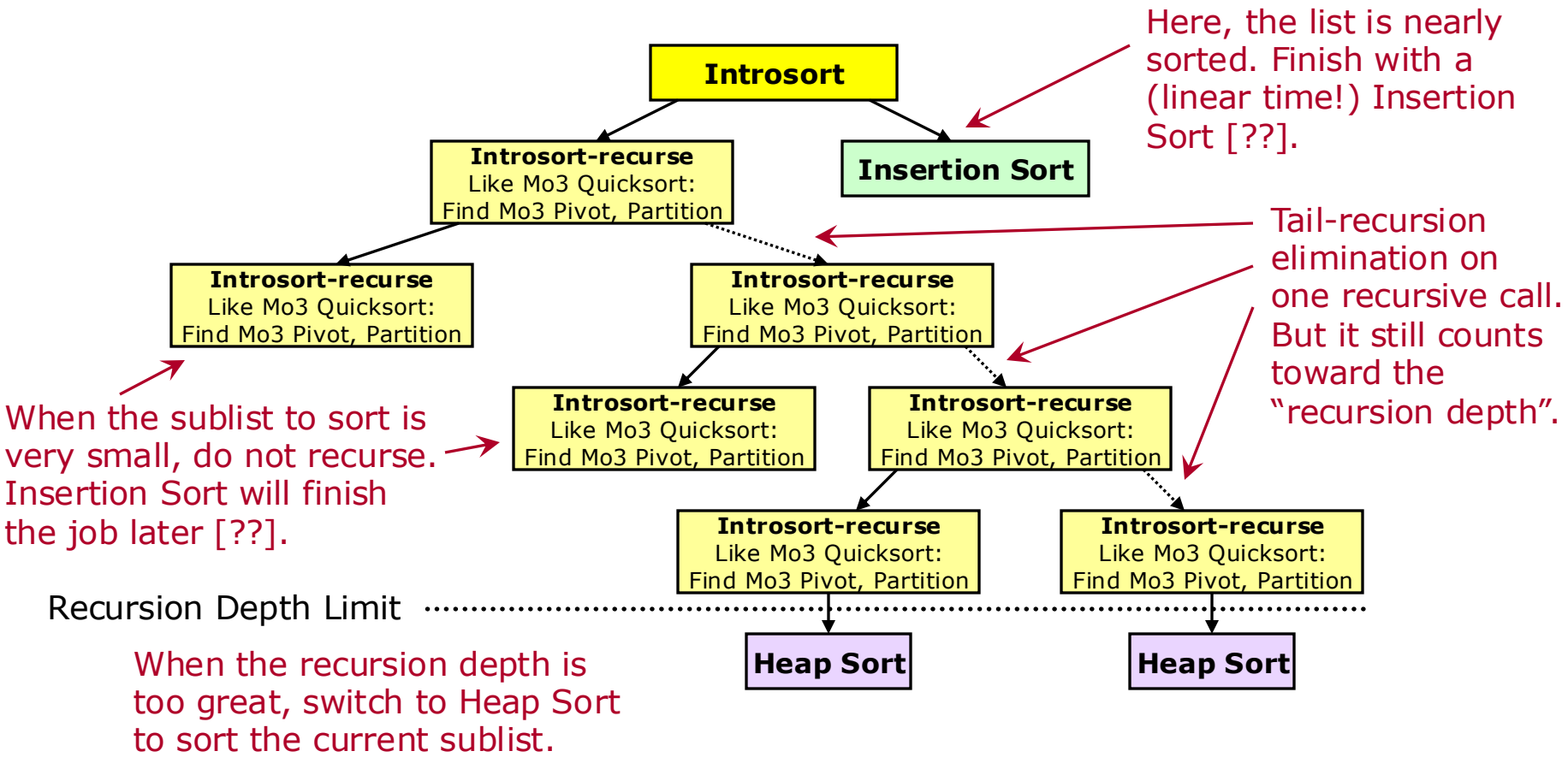
Musser's 1997 paper recommends the optimizations we covered:

- Median-of-3 pivot selection.
- Tail-recursion elimination on one recursive call.
- Stop the recursion prematurely, and finish with Insertion Sort. (*Maybe*. This can adversely affect cache performance.)

Introsort — Diagram

Here is an illustration of how Introsort works.

- In practice, the recursion will be much deeper than this.
- We *might* not do the Insertion Sort, due to its effect on cache hits.



Introsort — Analysis

Efficiency ☺☺

- Introsort is Θ(*n* log *n*).
- Introsort also has an average-case time of Θ(*n* log *n*)—of course.
  - Its average-case time is just as good as Quicksort. ☺☺

Requirements on Data ☹

- Introsort requires random-access data.

Space Usage 😐

- Introsort uses space for recursion.
  - Additional space: Θ(log *n*)—even if all recursion is eliminated.
  - This additional space need not hold any data items.

Stability ☹

- Introsort is not stable.

Performance on Nearly Sorted Data 😐

- Introsort is not significantly faster or slower on nearly sorted data.

Introsort — Notes

Our discussion of Quicksort & Introsort might suggest that their average-case time is significantly better than Merge Sort.

Historically, this has been largely the case. However, on modern architectures, a well optimized Merge Sort *can* be faster.

This is a tricky issue. Relative speed depends on:

- The processor used, and the performance of its cache.
- The type of the data being sorted.
- The data structure used, and its size.

It appears to me [GGC] that, in practice, use of the Quicksort family of algorithms—including Introsort—is fading.

For example, the old C Standard Library function `qsort` traditionally did Quicksort (thus the name). But at least one major implementation now uses Merge Sort in this function.