Collocation approximation of the monodromy operator of periodic, linear DDEs

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Outline

1. stability chart examples from machining
2. abstractions for periodic, linear DDEs
3. Chebyshev collocation
4. estimates for ODE initial value problems
5. approximation of the monodromy operator
Example 1: Turning

1 DOF model for regenerative vibrations of cutting tool with mass $m$, stiffness $k$, and damping $c$:

\[ m\ddot{x} + c\dot{x} + kx = \Delta F_x \]

$\Delta F_x = \Delta F_x(f)$ is $x$-component of cutting force variation, fcn of chip thickness $f$. Linearizing at prescribed thickness $f_0$ gives (for $k_1$ is constant)\(^a\)

\[ \Delta F_x \approx k_1(x(t - \tau) - x(t)) \]

\(^a\tau\) is rotation time of workpiece; $\Omega = 60/\tau$ is rot. rate (RPM)
QUESTION: Suppose $m, c, k$ are fixed. For which values $\Omega, k_1$ is this turning DDE (linearly) stable\textsuperscript{a}?

\textsuperscript{a}Definition. A linear, homogeneous DDE is stable (i.e. asymptotically stable) if all solutions decay to zero.
Example 1: Turning stability chart

(Based on 150 × 150 points in parameter plane. Compare to exact chart. For $\Omega \gtrsim 1000$, boundary comes within one point of correct. For $\Omega \lesssim 1000$, problem is stiffness, below.)
Example 2: (Interrupted) milling

1 DOF linearized model for regenerative vibrations:

\[ m \ddot{x} + c \dot{x} + k x = w h(t)(x(t - \tau) - x(t)) \]

But \( h(t) \) has the following nonsmooth, time-dependent form:

\[
\begin{array}{c}
\text{QUESTION: Suppose } m, c, k \text{ all fixed. For which values } \\
\Omega = 60/\tau, w \text{ is this milling DDE stable?}
\end{array}
\]
Example 2: Milling stability chart

(Compare to Insperger, et al., *Multiple chatter frequencies in milling processes*, J. Sound Vibration (2003).)
Conventions

- We consider \textit{linear, periodic-coefficient DDEs with fixed delays}. We assume rational relations among delays and coefficient periods. (For this talk: only one delay and period=delay.)

- Put in standard first-order form
  
  \[ \dot{y}(t) = A(t, \epsilon)y(t) + B(t, \epsilon)y(t - \tau) \]

  where \( A, B \) have \( \tau \)-periodic dependence on \( t \) and depend continuously on parameters \( \epsilon \in \mathbb{R}^d \) (typically \( d = 1, 2, 3 \)).

- We assume \( A, B \) are \textit{piecewise analytic} functions of \( t \).
Our Mission

- Construct a fast and accurate numerical method (based on *Chebyshev collocation*, below) for stability charts for linear, periodic DDE problems with piecewise-analytic coefficients.

- Prove it works. (Prove estimates for accuracy of IVP solutions. Prove estimates for eigenvalues.)

- Build an easy to use MATLAB package to implement it.

(STATUS July 2004: Mostly done including estimates (for constant non-delayed-coefficient cases). MATLAB suite in early version. See website www.cs.uaf.edu/~bueler/DDEcharts.htm.)
Recall (for linear, periodic DDE)

Initial value problem

\[ \dot{y} = A(t)y + B(t)y_{-\tau}, \quad y(t) = \phi(t) \text{ for } t \in [-\tau, 0] \]

has solution (monodromy; delayed FTM):

\[ (U\mathbf{f})(t) = \Phi(t) \left[ \mathbf{f}(1) + \int_{-1}^{t} \Phi^{-1}(s)B(s)\mathbf{f}(s) \, ds \right] \]

(where \( \dot{\Phi} = A(t)\Phi, \Phi(0) = I \)).

Soln of IVP:

\[ y_{n+1} = Uy_n, \quad y_0 = \phi \]
Abstract view of linear, periodic DDE

$U$ is a compact operator on $C([0, \tau])$.

Our class of DDE are simply linear difference eqns with compact generator in $C([0, \tau])$: $y_{n+1} = Uy_n$.

Compact ops are (norm-)limits of finite rank operators.

Stability: $\rho(U) < 1$ if and only if DDE is stable.\(^b\)

\(^a\)It is formed from an integral operator and $f \mapsto f(1)$, a finite rank operator.

\(^b\)Caveat: this is eigenvalue stability. Degree of nonnormality of $U$ does matter.
Chebyshev poly approx: 3 good reasons

- Polynomial and Fourier approximation (“spectral approximation”) converges faster than finite diff or finite elem or cubic splines or wavelets on analytic functions.

- Though the coefficients in our DDE are periodic the solutions are not. Thus Fourier not so good. (Also: poly approx can be good on each piece of a piecewise-analytic fcn without generating Gibbs phenomena.)

- Chebyshev points are nearly optimal polynomial interpolation points for minimizing uniform error.
Chebyshev collocation points

Chebyshev poly approx can be implemented by *collocation*. For degree $N$, Cheb collocation points are

$$t_j = \cos\left(j \frac{\pi}{N}\right), \quad j = 0, \ldots, N.$$  

($t_j$ are projections of equally-spaced points on unit circle$^a$).

Note $t_j \in [-1, 1]$. (If needed, shift the $t_j$ to interval $[0, \tau]$.)

$^a$The *Fourier* collocation points. Cheb collocation can be implemented by FFT.
Cheb spectral differentiation

1. Given $f(t)$ on $[-1, 1]$.
2. Construct interpolating polynomial $p(t)$: $p(t_j) = f(t_j)$.
3. Find $\dot{p}$.
4. Evaluate it at $t_j$: $\dot{f}(t_j) \approx \dot{p}(t_j)$.

This gives a matrix approximation of derivative $\frac{d}{dt}$:

$$\dot{f} \approx \dot{p} \quad \text{(represented by)} \quad D_N \nu$$
Cheb collocation approx of $U$

Use Cheb matrix approximations: (i) $D_N \approx \frac{d}{dt}$ (of a vector-valued fcn); (ii) $M_A \approx$ (mult by $A(t)$); (iii) $M_B \approx$ (mult by $B(t)$). Modify these to incorporate ODE initial condition: $y(0) = \phi(0)$.

\[
\dot{y} = A(t)y + B(t)y_{-\tau} \text{ with } y(t) = \phi(t), \ t \in [-\tau, 0]
\]

is approximated by

\[
D_N v = M_A v + M_B w
\]

(here $v \approx y$, $w \approx \phi$).

Solving for $v$ is approximating $U$:

\[
U \approx U_N \equiv (D_N - M_A)^{-1} M_B.
\]
Consider scalar DDE: $\dot{x} = -x + \frac{1}{2}x_{-2}$ with $N = 3$. Then $D_N$, $M_A$, $M_B$, and $U_N = (D_N - M_A)^{-1}M_B$ are $4 \times 4$ matrices. Last rows modified to enforce initial condition. $D_N$, $U_N$ generally dense.

\[
M_A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad M_B = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{pmatrix}
\]

\[
D_N = \begin{pmatrix} 19/6 & -4 & 4/3 & -1/2 \\ 1 & -1/3 & -1 & 1/3 \\ -1/3 & 1 & 1/3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_N = \begin{pmatrix} 0.2058 & 0.2469 & 0.1152 & 0 \\ 0.1852 & 0.2222 & 0.2037 & 0 \\ 0.6626 & -0.1049 & 0.2510 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]
Example: Eigs of a scalar DDE

For $\dot{x} = -x + (1/2)x_{-1}$ we compare $U_N$ eigenvalues to exact:

“Exact” method: Each root $\mu \in \mathbb{C}$ of characteristic eqn $\mu = -1 + 0.5e^{-\mu}$ is eigenvalue of $U$. Reduce char eqn to real variable problem. Solve by robust one-variable method (e.g. bisection) to $10^{-14}$ relative accuracy.

vs

*Cheb collocation with $N = 29$: Compute $U_N$. Find eigs of $U_N$.*

RESULT: Largest 7 eigenvalues of $U_N$ are each accurate to more than 12 digits.
Example, cont

For remaining 23 eigenvalues, here’s the picture:

SUMMARY: Over 100 digits of correct eigenvalues from $30 \times 30$ matrix approx of $U$.
Only eigs near $0 \in \mathbb{C}$ are inaccurate (irrelevant for stability).
Cost of a stability chart

Using numerical method to produce $m \times m$ approximation to $U$, the time to produce a chart is

$$O((\# \text{ of pixels}) \cdot m^3)$$

with standard estimates on QR method for eigenvalues.

$m$ matters! Small is good!
Accuracy of Chebyshev interpolation

*Theorem* [classical]. Let $p$ be degree $N$ poly for $f$ using $N + 1$ Cheb colloc pts. If $f$ analytic in a $\mathcal{C}$-neighborhood $R$ of $[-1, 1]$ then there exists $C$ s. t.

$$\|f - p\|_{\infty} \leq C(S + s)^{-N}$$

where $S, s$ are semi axes of ellipse $E$ s. t. $[-1, 1] \subset E \subset R$.

Moral: If $f$ analytic then $p$ improves by a fixed number of digits per increase by one in $N$. 
Theorem. Consider IVP \( \dot{y} = ay + b(t)y_{-\tau}, y(t) = \phi(t) \) for \( t \in [-\tau, 0] \). Let \( q \) be the interpolating poly of delayed term \( b\phi \). Find degree \( N \) collocation solution \( p(t) \), a polynomial. Then

\[
\| y - p \|_\infty \leq c_1\| q - b\phi \|_\infty + c_2|\dot{p}(0) - a\phi(0) - b(0)\phi(-\tau)|.
\]

\( c_1, c_2 \) depend on \( a \) but are \( O(1) \) in \( N \).

Thus

- Error has two sources: (i) interpolation error for delayed term; (ii) residual error at initial time from difficulty of nonhomogeneous ODE problem.

- \textit{a posteriori} result: Do computation, get \textit{proven} estimate of quality of solution based on result.
Example: accuracy in DDE IVP

Find $y(t)$ on $[0, 2]$ if $\dot{y} = 3y + (t - 1)y_{-2}$, $\phi(t) = 1$. 

![Graph showing estimate and actual error for different N values]
Estimates for eigenvalues of $U$

In basis of Chebyshev polynomials $\{T_j\}$, matrix entries of $U$ on $C([-1, 1])$ can be computed by inner products: $U_{jk} = \langle T_j, UT_k \rangle$.

Note $y = UT_k$ is the solution of an IVP. We use previous $a posteriori$ estimate to show $\|UT_k - (U_N)T_k\|$ small$^a$ for $k$ up to about $\frac{3}{4}N$.

Now use eigenvalue perturbation theory$^b$ to show large eigenvalues of $U_N$ are close to those of $U$.

$^a$Recall $U_N$ is Chebyshev approximation to $U$.

$^b$An extension of the Bauer-Fike theorem to compact operators on Hilbert spaces; need to transfer $U$ to act on Sobolev space $H^1_{Cheb}$. 
Provable eigenvalues of $U$.

**Example:** Consider $\dot{y} = -2y + (1 + \sin(3\pi t))y_{-2}$. Let $N = 95$.

**Result:** Dots are eigs of $U_N$; discs are *proven* error bounds for sufficiently large eigs of $U$. (If $\mu$ is an eig of $U$ and $|\mu| \geq 0.2$ then $\mu$ is in one of these discs.) This DDE is *proven stable*.

Size of discs drops exponentially with increasing $N \gtrsim 90$ (this example).
Why one really cares about $U$

The interesting systems are nonlinear DDEs. The linear, periodic DDEs are just their linearizations.

Questions about nonlinear DDE:
- find fixed points and periodic orbits
- nature of bifurcations?

To study the latter question we need good bases for spaces of stable and unstable directions. Good approximation to $U$ means good bases for these purposes.

But that’s another talk . . .