I. LIST OF PUTNAM PROBLEMS: A1’s, A2’s, B1’s, AND B2’s

(1998) A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side length of the cube?

(1998) A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that A + B depends only on the arc length, and not on the position, of s.

(1998) B1. Find the minimum value of
\[ \left( x + \frac{1}{x} \right)^6 - \left( x^6 + \frac{1}{x^6} \right) - 2 \]
for \( x > 0 \).

(1998) B2. Given a point \((a, b)\) with \(0 < b < a\), determine the minimum perimeter of a triangle with one vertex at \((a, b)\), one on the x-axis, and one on the line \(y = x\). You may assume that a triangle of minimum perimeter exists.

(1989) A1. How many primes among the positive integers, written as usual in base 10, are alternating 1’s and 0’s, beginning and ending with 1?

(1989) A2. Evaluate \( \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx \) where \( a \) and \( b \) are positive.

(1989) B1. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express you answer in the form \( \frac{a\sqrt{b} + c}{d} \), where \( a, b, c, d \) are integers.

(1989) B2. Let \( S \) be a non-empty set with an associative operation that is left and right cancellative (\( xy = xz \) implies \( y = z \), and \( yx = zx \) implies \( y = z \)). Assume that for every \( a \) in \( S \) the set \( \{a^n : n = 1, 2, 3, \ldots\} \) is finite. Must \( S \) be a group?

(1977) B1. Evaluate the infinite product
\[ \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}. \]

(1977) B2. Given a convex quadrilateral \(ABCD\) and a point \(O\) not in the plane of \(ABCD\), locate point \(A'\) on line \(OA\), point \(B'\) on line \(OB\), point \(C'\) on line \(OC\), and point \(D'\) on line \(OD\) so that \(A'B'C'D'\) is a parallelogram.