
Let $N$ be a compact Riemannian manifold and let $h$ be a Morse-Bott function on $N$, that is, the critical points of $h$ form a submanifold $M$ and in certain local coordinates $h$ near $M$ is quadratic.

E. Witten famously used $h$ to modify the exterior derivative and Hodge Laplacian on $N$: $d(\alpha) = e^{-\alpha h}d\alpha$ and $L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha)$. Because $L(\alpha)$ for large $\alpha$ is a Schrödinger operator with a strongly confining potential, with $M$ at the bottom of the wells, the eigenforms of $L(\alpha)$ concentrate near $M$.

This paper proves that as $\alpha \to \infty$ the eigenvalues of $L(\alpha)$ either diverge or converge to the eigenvalues of the Hodge Laplacian $\Delta$ on $M$, and estimates the rate of convergence. More precisely, $\Delta$ acts on forms with values in the orientation bundle of $E$, the normal bundle of $M$. In fact, this paper further generalizes to $L(\alpha)$ acting on forms with values in a flat vector bundle.

The eigenvalues of $L(\alpha)$ which are bounded as $\alpha \to \infty$ converge to the eigenvalues of $\Delta$ at a rate at least $\alpha^{-1/2}$ as $\alpha \to \infty$. This estimate follows from the adiabatic limit which expands the metric in directions orthogonal to the fibers of $E \to E^-$, where $E^-$ is the subbundle of $E$ corresponding to the negative quadratic directions of $h$. It is necessary to compare $L(\alpha)$ to a Witten Laplacian $\Box(\alpha)$ living entirely on $E$ and to solve the Hodge theory of $\Box(\alpha)$.

As corollaries the current paper gives a new analytic proof of the Morse-Bott inequalities, proven from probabilistic analysis of the Witten Laplacian by J.-M. Bismut [J. Differential Geom. 23 (1986), no. 3, 207–240; MR 87m:58169], and of the Thom isomorphism.

Edward L. Bueler (1-AK)