Poisson’s equation by the FEM, continued:

general boundary conditions and error analysis

This note extends [2]. It assumes the reader has absorbed chapter one of [5].

Consider the boundary value problem described in the classical manner as

\[(1) \quad -\Delta u = f \quad \text{on} \quad D \subset \mathbb{R}^2, \quad u = g_D \quad \text{on} \quad \Gamma_D, \quad \text{and} \quad \frac{\partial u}{\partial n} = g_N \quad \text{on} \quad \Gamma_N\]

where \(\partial D\) is the boundary of an open, connected region \(D\). Assume that the subsets \(\Gamma_D, \Gamma_N \subset \partial D\) are disjoint, \(\Gamma_D \cup \Gamma_N = \partial D\), and \(\frac{\partial u}{\partial n}\) is the directional derivative of \(u\) in the outward direction normal to \(\partial D\). Suppose \(\partial D\) is a smooth or piecewise-smooth closed, continuous curve. Furthermore, assume \(f, g_D, g_N\) are well-behaved enough so that we do not worry about them; concretely, suppose \(\Gamma_N\) is open, \(f\) is continuous on \(D\) and \(g_D, g_N\) are continuous and bounded on \(\Gamma_D, \Gamma_N\) respectively.

Now we seek a variational formulation of (1). Let \(v\) be a function in a (for now) unspecified space. Multiply \(f = -\Delta u\) by \(v\) and integrate:

\[
\int_D fv = -\int_D v\Delta u = \int_D \nabla v \cdot \nabla u - \int_{\partial D} v \frac{\partial u}{\partial n} = \int_D \nabla v \cdot \nabla u - \int_{\Gamma_D} v \frac{\partial u}{\partial n} - \int_{\Gamma_N} vg_N,
\]

by Green’s theorem. Of the two boundary terms on the right, the second is clearly desirable because it brings \(g_N\) into the variational formulation. On the other hand, and based on the pure Dirichlet problem for (1) addressed in [2], we assume \(v\) is zero on \(\Gamma_D\) in order to eliminate that boundary term.

I now introduce not-completely-standard, but good(!) notation because we need to talk about functions which are specified on a piece of the boundary of \(D\).

**Definition.** Suppose \(\Gamma \subset \partial D\) is measurable. Let

\[H^1_\varphi(\Gamma) = \{ v \text{ measurable on } D \mid v \in L_2(D), \nabla v \in L_2(D), \ v = \varphi \text{ on } \Gamma \}.\]

Comparing to existing notation, \(H^1_0(D)\) becomes \(H^1_0(\partial D)\) in this notation. Note \(H^1_\varphi(\Gamma) \subset H^1(D)\) is a vector subspace only if \(\varphi = 0\). If \(\varphi \neq 0\) then it is an affine space with \(v - w \in H^1(\Gamma)\) if \(v, w \in H^1_\varphi(\Gamma)\). Finally, in this definition, as in all of [5] as well, the mathematical issue of traces is ignored. A “trace operator” is the precise way to define boundary values of a \(H^1(D)\) function; see [3, section 5.5] or [1, section 1.6].

The variational formulation of (1) is to find \(u \in H^1_{g_D}(\Gamma_D)\) such that

\[(2) \quad \int_D \nabla u \cdot \nabla v = \int_D fv + \int_{\Gamma_N} g_N v, \quad \text{for all} \quad v \in H^1_0(\Gamma_D).\]

That is, we require \(u\) to take on the desired values \(g_D\) on \(\Gamma_D\) while test functions \(v\) are zero on \(\Gamma_D\). If \(\Gamma_D = \emptyset\) then a solution to (1) or (2) is not unique as an arbitrary constant can be added to any solution to get a new one.
Before formulating a finite element version of (2), we make two additional assumptions on the geometry of the domain and its boundaries. First, we require that $D$ is in fact a polygonal domain with vertices (corners) $q_s, s = 1, \ldots, M$. Secondly, we suppose $\Gamma_D$ is a union of closed edges of $\partial D$, that is, the “transition” from $\Gamma_D$ to $\Gamma_N$ occurs only at vertices of the polygon $D$. These assumptions facilitate error analysis in the finite element method.

What is the finite element method here, how accurate is it, and how to compute it in practice? We restrict ourselves to the simplest case of piecewise linear finite elements on a triangulation. Suppose the triangulation $T$ on $D$ has nodes $p_j$ for $j = 1, \ldots, N_p$. Furthermore, suppose the set of nodes $\{p_j\}$ includes all the vertices $q_s$. Also, suppose that the first $N$ nodes ($N \leq N_p$) are either in the interior of $D$ or are in $\Gamma_N$ and that nodes $p_{N+1}, \ldots, p_{N_p}$ are in $\Gamma_D$. The values of $u$ at the nodes $p_1, \ldots, p_N$ are the unknowns.

Let $V_h$ be the space of continuous functions which are linear on each $T \in T$ and which are zero on $\Gamma_D$, so that

$$V_h \subset H^1_D(\Gamma_D).$$

For $j = 1, \ldots, N_p$, let $\varphi_j$ be the “hat” function on $p_j$, that is, $\varphi_j$ is continuous, piecewise linear, and satisfies $\varphi_j(p_k) = \delta_{jk}$. For $j = 1, \ldots, N$, $\varphi_j \in V_h$. Furthermore, $\{\varphi_j\}_{j=1}^N$ is a basis of $V_h$.

The finite element formulation is to seek real coefficients $\xi_j, j = 1, \ldots, N$, for the approximate solution $u_h$,

$$u_h = \sum_{j=1}^N \xi_j \varphi_j + \sum_{j=N+1}^{N_p} g_D(p_j) \varphi_j,$$

so that

$$\int_D \nabla u_h \cdot \nabla v = \int_D f v + \int_{\Gamma_N} g_N v \quad \text{for all } v \in V_h,$$

that is, so that

$$\int_D \nabla u_h \cdot \nabla \varphi_k = \int_D f \varphi_k + \int_{\Gamma_N} g_N \varphi_k \quad \text{for all } k = 1, \ldots, N.$$

Note that $u_h \in H^1(D)$ and $u_h|_{\Gamma_D}$ is a piecewise linear approximation to $g_D$ along $\Gamma_D$, and thus “$u_h \in H^1_D(\Gamma_D)$” generally only in an approximate sense.

It follows, by substitution of (3) into (4), that we will solve a linear system $Ax = b$ where

$$a_{kj} = \sum_{j=1}^N \int_D \nabla \varphi_j \cdot \nabla \varphi_k \quad \text{and} \quad b_k = \int_D f \varphi_k + \int_{\Gamma_N} g_N \varphi_k - \sum_{j=N+1}^{N_p} g_D(p_j) \int_D \nabla \varphi_j \cdot \nabla \varphi_k.$$

Note the presence of stiffness terms on the right side.\(^1\)

\(^1\)One may also formulate the problem with additional trivial equations so as to have all stiffness terms on the left-hand-side.
Though existence is not a particular concern here, \( A \) is symmetric and positive definite. In fact, if \( w = \sum_{j=1}^{N} y_j \varphi_j \in V_h \) is nonzero and at least one point \( p_j \in \Gamma_D \) then \( 0 < \int_{D} |\nabla w|^2 = y^\top A y \) where \( y = (y_j) \). Thus we have unique existence for the discrete problem (3) and (4).

**Code.** Here is an implementation which barely fits on one page. As in [2], the integrals \( \int_{D} f \varphi_k \) are computed by first approximating \( f \) by a piecewise linear function. Similarly, the boundary integrals \( \int_{\Gamma_N} g_N \varphi_k \) are done by Simpson’s rule on each edge of the triangulation which is in \( \Gamma_N \). Note the inclusion of a helper function \( \text{Ncntrb} \) which computes Neumann boundary contributions to the load vector. Many further details of the implementation are addressed for the pure Dirichlet case \((\Gamma_N = \emptyset, g_D \equiv 0)\) in [2].

```matlab
function [uh,un]=poissonDN(f,gD,gN,fd,fGam,h0,p,t)
%POISSONDN Solve Poisson’s equation on (open) domain D by the FE method ...

geps=.001*h0; int=(feval(fd,p) < -geps);  % int true if node in interior
inGamD=(feval(fGam,p) < +geps)&(~int);  % inGamD true if node in GamD
inGamN=~(int|inGamD);  % inGamN true if node in GamN
if sum(inGamD)==0, error('no unique soln; needs Dirichlet points'), end
Np=size(p,1); uh=zeros(Np,1); ff=uh; un=uh;  % Np=total # of nodes
N=sum(~inGamD); un(~inGamD)=(1:N)';  % N=# of unknowns
N=un('inGamN')=(1:N)';  % N=# of unknowns
for j=1:Np % eval f once for each node; fill in known bdry vals on Gam1
ff(j)=feval(f,p(j,:));
end

% loop over triangles to set up stiffness matrix A and load vector b
A=sparse(N,N); b=zeros(N,1);
for n=1:size(t,1)
    j=t(n,1); k=t(n,2); l=t(n,3); vj=un(j); vk=un(k); vl=un(l);
    J=[p(k,1)-p(j,1), p(l,1)-p(j,1); p(k,2)-p(j,2), p(l,2)-p(j,2)];
    ar=abs(det(J))/2; C=ar/12; Q=inv(J'*J);
    fT=[ff(j) ff(k) ff(l)];
    if vj>0
        A(vj,vj)=A(vj,vj)+ar*sum(sum(Q)); b(vj)=b(vj)+C*fT*[2 1 1]';
    end
    if vk>0
        A(vj,vk)=A(vj,vk)-ar*sum(Q(:,1)); A(vk,vj)=A(vj,vk);
    end
    if vl>0
        A(vj,vl)=A(vj,vl)-ar*sum(Q(:,2)); A(vl,vj)=A(vj,vl);
    end
else % pj in GamD
    if vk>0, b(vk)=b(vk)+uh(j)*ar*sum(Q(:,1)); end
    if vl>0, b(vl)=b(vl)+uh(j)*ar*sum(Q(:,2)); end, end
else % pk in GamD
    if vj>0, b(vj)=b(vj)+uh(k)*ar*sum(Q(:,1)); end
    if vl>0, b(vl)=b(vl)-uh(k)*ar*Q(1,2); end, end
else % pl in Gam1
   b(vl)=b(vl)+C*fT*[1 1 2]';
end
end
```

if vj>0, b(vj)=b(vj)+uh(l)*ar*sum(Q(:,2)); end
if vk>0, b(vk)=b(vk)-uh(l)*ar*Q(1,2); end, end
% now add Neumann contribution
if inGamN(j)
    if ~int(k), b(vj)=b(vj)+Ncntrb(gN,p(j,:),p(k,:)); end
    if ~int(l), b(vj)=b(vj)+Ncntrb(gN,p(j,:),p(l,:)); end,
end
if inGamN(k)
    if ~int(j), b(vk)=b(vk)+Ncntrb(gN,p(k,:),p(j,:)); end
    if ~int(l), b(vk)=b(vk)+Ncntrb(gN,p(k,:),p(l,:)); end
end
if inGamN(l)
    if ~int(j), b(vl)=b(vl)+Ncntrb(gN,p(l,:),p(j,:)); end
    if ~int(k), b(vl)=b(vl)+Ncntrb(gN,p(l,:),p(k,:)); end,
end
end

uh(~inGamD)=A\b; % solve for FE solution
trimesh(t,p(:,1),p(:,2),uh), axis tight % display

function w=Ncntrb(gN,p,q); % compute Neumann contribution by Simpson’s rule
w=norm(p-q)*( feval(gN,p) + 2*feval(gN,(p+q)/2) )/6;

Example 1. For a first verification, I consider a problem for which I know that the exact solution is in $H^2(D)$, namely,

$$-\triangle u = 0 \quad \text{on } D = [0,1] \times [0,1], \quad \text{and} \quad u = x^2 - y^2 \quad \text{on } \Gamma_D = \partial D.$$ 

In particular, $f \equiv 0$ and $\Gamma_N = \emptyset$ so this is a classical Laplace equation example. The solution on $D$ is, of course, $u(x,y) = x^2 - y^2$.

To use distmesh2d.m and poissonDN.m on this problem, we choose a mesh with typical feature size of $h_0 = 0.1$:

```
>> fd=inline('drectangle(p,0,1,0,1)', 'p'); fGam=inline('-1', 'p');
>> h0=0.1; [p,t]=distmesh2d(fd,@huniform,h0,[0,0;1,1],[0,0;0,1;1,0;1,1]);
>> f=inline('0', 'p'); gD=inline('p(:,1).^2-p(:,2).^2', 'p');
>> [uh,un]=poissonDN(f,gD,f,fGam,h0,p,t); err=max(abs(uh-gD(p)))
```

The result is an error of $\text{err} = 5.1 \times 10^{-4}$. In fact, we consider $h_0(k) = 0.5(3/5)^k$ for $k = 0,1,\ldots,6$ on the same problem. The picture of convergence is in figure 1. It seems we have $O(h_0^2)$ convergence which is optimal.

Let us now attempt to prove a theorem on error. Recall that $\|f\|_1 = \int_D |\nabla f|^2 = \sum_{i=1}^2 \int_D |f_{x_i}|^2$ and $\|f\|_2 = \sum_{j=1}^2 \int_D |f_{x_j,x_j}|^2$ while $\|f\|_{L^2} = \int_D |f|^2$ and $\|f\|_{H^1} = \|f\|_{L^2} + \|\nabla f\|_{L^2}$. That is, $\| \cdot \|_{L^2}$ and $\| \cdot \|_{H^1}$ are norms while $| \cdot |_1$ and $| \cdot |_2$ are semi-norms.

**Theorem 1.** Suppose $u \in H^1_{g_D}(\Gamma_D)$ solves the variational problem (2) and that $u_h$ satisfies (3) and (4) for a fixed triangulation $T$. Let $\gamma = \sum_{j=N+1}^{N_p} g_D(p_j)\varphi_j \in H^1(D)$. Then

$$|u - u_h|_1 \leq |u - (v + \gamma)|_1$$

for all $v \in V_h$. In particular, suppose $T$ is a regular triangulation with $h > 0$ a bound on the maximum side length (“diameter”) of triangles $T \in T$ and with $B > 0$ a bound
on the ratios $\frac{h}{\rho}$ of the diameter $h_T$ to the radius $\rho_T$ of the largest inscribed circle inside $T \in \mathcal{T}$. It follows that

$$|u - u_h|_1 \leq CBh|u|_2$$

for some $C$ depending only on $D$.

**Proof.** By subtraction of (4) from (2) we have

$$\int_D \nabla (u - u_h) \cdot \nabla v = 0 \quad \text{for all } v \in V_h.$$ 

In particular, $\int_D \nabla (u - u_h) \cdot (u_h - \gamma - v) = 0$ if $v \in V_h$. Thus

$$|u - u_h|^2 = \int_D \nabla (u - u_h) \cdot \nabla (u - u_h + u_h - \gamma - v) = \int_D \nabla (u - u_h) \cdot \nabla (u - \gamma - v) \leq |u - u_h|_1 |u - (v + \gamma)|_1,$$

by Cauchy-Schwarz. Estimate (5) follows.

We now use the theory of interpolation error in section 4.2 of [5]. Estimate (4.17) applies, namely $|u - \pi_h u|_1 \leq CBh|u|_2$, where $\pi_h u$ is the piecewise-linear interpolant of $u$, and furthermore $\pi_h u = v + \gamma$ for $v \in V_h$. Thus (6) follows from (5). \qed

To make this result useful we need to know that $| \cdot |_1$ controls $\| \cdot \|_{H^1}$ when we consider functions which are zero on a “significant” part of $\partial D$.

**Lemma 2** (generalization of Poincaré’s inequality; proposition (5.3.3) in [1]). Suppose $D$ is a union of domains that each are star-shaped with respect to finitely-many discs (see [1]), or, for example, suppose $D$ is a polygon. Suppose $\Gamma_D \subset \partial D$ is closed and that $\text{meas}(\Gamma_D) > 0$ (length$(\Gamma_D) > 0$ if $\Gamma_D$ is a countable union of smooth curve segments). There exists $C > 0$ depending only on $D$ and $\Gamma_D$ such that

$$\|v\|_{H^1} \leq C|v|_1 \quad \text{for all } v \in H^1_0(\Gamma_D).$$
Thus we have the following corollary to theorem 1.

**Corollary 3.** Assuming \( u, u_h, T, \) and \( \gamma \) are as in theorem 1, and assume \( D \) is (for example) a polygon. Suppose \( \epsilon \in H^1(D) \) is some function such that \( \epsilon|_{\Gamma_D} = g_D - \gamma|_{\Gamma_D} = u|_{\Gamma_D} - u_h|_{\Gamma_D} \). Then
\[
\|u - u_h\|_{H^1} \leq C_1 B h |u|_2 + C_2 \|\epsilon\|_{H^1}
\]
for \( C_i \) depending only on \( D \) and \( \Gamma_D \).

**Proof.** Note \( u - u_h - \epsilon \in H^1_0(\Gamma_D) \). Thus from lemma 2 and theorem 1,
\[
\|u - u_h\|_{H^1} \leq \|u - u_h - \epsilon\|_{H^1} + \|\epsilon\|_{H^1} \leq C |u - u_h - \epsilon|_1 + \|\epsilon\|_{H^1} \\
\leq C |u - u_h|_1 + (C + 1) \|\epsilon\|_{H^1} \leq CC' B h |u|_2 + (C + 1) \|\epsilon\|_{H^1}.
\]

\( \square \)

In fact \( \epsilon \) can be chosen small if \( g_D \) is well-approximated by piecewise linear functions.

**Exercise.** (a) Assume \( g_D \) is continuous and has bounded second derivative on each segment of \( \Gamma_D \); recall we assume \( D \) is a polygon. Consider a triangle \( T \) for which one edge is contained in the Dirichlet boundary \( \Gamma_D \). In particular, put coordinates on \( T \) and suppose \( T \) has dimensions \( \delta, d, \omega \) as shown in figure 2; the \( x = 0 \) edge is in \( \Gamma_D \). Let \( \psi(y) = g_D - \gamma|_{\Gamma_D} \) so \( \psi(0) = \psi(\delta) = 0 \). Define
\[
\epsilon(x, y) = \frac{d - x}{d} \psi \left( \frac{dy - \omega x}{d - x} \right).
\]
Show that
\[
\int_T |\epsilon|^2 \leq \|\psi\|^2_{\infty} \frac{\delta d}{3} \quad \text{and} \quad \int_T |\nabla \epsilon|^2 \leq \|\psi\|^2_{\infty} \frac{\delta}{d} + \|\psi'|^2_{\infty} \int_T \left( 2 \left( \frac{y - \omega}{d - x} \right)^2 + 1 \right) dx dy.
\]
(b) Show that
\[
-\frac{\omega}{d} \leq \frac{y - \omega}{d - x} \leq \frac{\delta - \omega}{d}
\]
if \( (x, y) \in T \). Now assume that the triangulation is regular so that the ratio \( h_T/\rho_T \) is bounded by \( B \) and so that diameters \( h_T \) are bounded by \( h \). Show there exists constants \( C_i > 0 \) so that
\[
\int_T |\epsilon|^2 \leq C_1 \|\psi\|^2_{\infty} h^2 \quad \text{and} \quad \int_T |\nabla \epsilon|^2 \leq C_2 \|\psi\|^2_{\infty} + C_3 \|\psi'|^2_{\infty} h^2.
\]
(c) By one-dimensional interpolation theory ([5, page 25]) \( \|\psi\|_{\infty} \leq \|g''_D\|_{\infty} \delta^2 / 8 \leq \|g''_D\|_{\infty} h^2 / 8 \) and \( \|\psi'|_{\infty} \leq \|g''_D\|_{\infty} \delta \leq \|g''_D\|_{\infty} h \) on each \( T \) which meets \( \Gamma_D \). Now define \( \epsilon \) on the entire triangulation as zero on each \( T \) which does not meet \( \Gamma_D \). Show that
\[
\|\epsilon\|_{L^2} \leq C_4 \|g''_D\|_{\infty} h^{5/2} \quad \text{and} \quad |\epsilon|_1 \leq C_5 \|g''_D\|_{\infty} h^{3/2}.
\]
(Hint: There are \( O(h^{-1}) \) triangles \( T \) along \( \Gamma_D \).)

From the above exercise we have another corollary of theorem 1.
Figure 2. A function $\epsilon(x,y)$ can be chosen to be small in the $\| \cdot \|_{H^1}$ sense on a boundary triangle $T$ which meets $\Gamma_D$ because $\psi = g_D - \gamma|_{\Gamma_D}$ is small on the boundary of $D$ if $g_D$ has bounded second derivative.

Corollary 4. Assuming $u, u_h, T$, and $\gamma$ are as in theorem 1, and assume $D$ is a polygon. Suppose $g_D$ is continuous on $\Gamma_D$ with bounded second derivative on each segment of $\Gamma_D$. Then

$$\|u - u_h\|_{H^1} \leq C_1 h |u|_2 + C_2 h^{3/2} \|g''_D\|_{\infty}$$

for $C_i$ depending only on $D$, $\Gamma_D$, and $B$.

This result is still unsatisfying, however, because it is possible that $u \notin H^2(D)$ for a boundary value problem of the form (1).

Example 2. Let $D = \{0 < x < \pi, 0 < y < 1\} \subset \mathbb{R}^2$ and consider the problem $\Delta u = 0$, $u(x,0) = u(0,y) = u(\pi,y) = u(x,1) = 0$. This is an example of problem (1) with $D$ a convex polygon, $f \equiv 0$, $\Gamma_D = \partial D$, $\Gamma_N = \emptyset$, and $g_D$ continuous. The solution, found by separation of variables and Fourier sine series, is

$$u(x,y) = \sum_{k=1}^{\infty} \frac{4 \sin(k\pi/2)}{\pi k^2} \sin(kx) \sinh(ky).$$

Is $u \in H^2(D)$? A picture of $u$, which “tents” to a point at $(x,y) = (\pi/2,1)$, will cause the reader to doubt it. In fact, the following MATLAB produces figure 3:

```matlab
>> N=40; y=0:.01:1; x=pi*y; [xx,yy]=meshgrid(x,y);
>> k=1:N; c=4*sin(k*pi/2)./(pi*k.^2.*sinh(k));
>> u=zeros(101); for k=1:N, u=u+c(k)*sin(k*xx).*sinh(k*yy); end
>> mesh(x,y,u), axis tight, xlabel x, ylabel y
```

Indeed, if $u^N$ is the partial sum then

$$\int_D (u^N_{xx})^2 = \frac{8}{\pi} \sum_{k=1}^{N} \frac{\sin^2(k\pi/2)}{\sinh^2 k} \int_0^1 \sinh^2(ky) \, dy = \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin^2(k\pi/2) \sinh(2k) - 2k}{k \cosh(2k) - 1}$$
using orthogonality \((\int_0^\pi \sin(jx) \sin(kx) \, dx = \frac{\pi}{2} \delta_{jk})\) and a few hyperbolic identities. But \(\frac{\sin(2k) - 2k}{\cosh(2k) - 1} \to 1\) as \(k \to \infty\) and also \(\sin^2(2j\pi/2) = 0\), so, by Bessel’s inequality,

\[
|u|^2 \geq \int_D (u_{xx}^N)^2 \geq c \sum_{j=1}^N \frac{1}{2j - 1},
\]

for \(c > 0\). The right-hand sum diverges as \(N \to \infty\). That is, \(u \notin H^2(D)\) even though \(u \in C^\infty(D)\). (Of course, \(u \notin C^\infty(\overline{D})\).)

![Figure 3. A solution \(u = u(x, y)\) to Laplace’s equation with continuous Dirichlet boundary values for which \(u\) is not in \(H^2(D)\).](image)

Now we report even worse news. That is, even if all the boundary data is zero, a transition from Dirichlet to Neumann boundary condition can cause a failure of \(H^2(D)\) regularity for solutions to Poisson’s equation.

**Example 3.** Let \(D\) be the upper half of the unit disc. Suppose \(\Gamma_D\) is the union of the part of \(\partial D\) which is on the positive real axis and the part of \(\partial D\) which is the upper unit circle. Thus \(\Gamma_N\) is the (open) part of \(\partial D\) which is on the negative real axis. Let

\[
u(r, \theta) = (1 - r^2)^{1/2} \sin(\theta/2)
\]
in polar coordinates. Note \(u = 0\) on \(\Gamma_D\) and \(\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \theta} = 0\) on \(\Gamma_N\). Furthermore

\[
\Delta u = -f(r, \theta) = -6r^{1/2} \sin(\theta/2),
\]

that is, \(u\) solves a Poisson equation \(-\Delta u = f\) where \(f\) is continuous on \(\overline{D}\) (and thus in \(L^2(D)\), of course). But, as the reader can check, \(u_{rr} \approx -\frac{1}{4} r^{-3/2}\) near \(r = 0\), and thus \(u_{rr} \notin L^2(D)\). Thus \(u \notin H^2(D)\).

Figure 4 shows this surface. It was generated by
I have used a triangular mesh to show the surface but there was no finite element calculation of $u$. For more on this example, see [1, section 5.5].

Figure 4. A solution $u = u(x, y)$ to Poisson’s equation $-\Delta u = f$ with $f$ continuous, and zero Dirichlet and Neumann boundary data, but for which $u$ is not in $H^2(D)$.

Thus warned, we recall the regularity result which does exist for Dirichlet boundary conditions. We use it, along with a duality argument [5, section 4.7], to give a final closer-to-optimal rate theorem on convergence of the FEM in the case of nonhomogeneous Dirichlet boundary conditions.

**Lemma 5.** (Elliptic regularity up to the boundary. See [3, section 6.3.2] and [4].) Suppose $\partial D$ is either $C^2$ or convex. Suppose $\Gamma_D$ is either empty or $\Gamma_D = \partial D$. There exists $C$ depending on $D$ and $\Gamma_D$ so that if $w$ solves $-\Delta w = f$, $w = 0$ on $\Gamma_D$, and $\frac{\partial w}{\partial n} = 0$ on $\Gamma_N = \partial D \setminus \Gamma_D$ then $|w|_2 \leq C \|f\|_{L^2}$.

**Theorem 6.** Suppose $D$ is a convex polygon and $\Gamma_D = \partial D$. Suppose $T$ is a regular triangulation of $D$ satisfying the hypotheses given on page 2 and the hypotheses of theorem 1. In particular, assume a bound $B$ on ratios $\rho_T/h_T$. There are constants $C_i$ depending only on $D$, $\Gamma_D$, and $B$ so that if $u$ solves (2) and $u_h$ solves (4) then

$$\|u - u_h\|_{L^2} \leq C_1 h^2 |u|_2 + C_2 h^{3/2} \|g_D''\|_{\infty}.$$

**Remark.** The proof below follows the common convention that the letter $C$ is a “generic positive constant” which absorbs positive combinations of previous constants.
Proof. Let \( e = u - u_h \) be the error. Consider the "dual problem"

\[-\Delta w = e, \quad w = 0 \text{ on } \partial D.\]

Elliptic regularity implies \( |w|_2 \leq C\|e\|_{L^2} \). Also, because \( \int_D \varphi e = -\int_D \varphi \Delta w = \int_D \nabla \varphi \cdot \nabla w - \int_{\partial D} \varphi \frac{\partial w}{\partial n} = \int_D \nabla \varphi \cdot \nabla w \) for all \( \varphi \in H_0^1(D) \), \( |w|_1^2 = \int_D we \leq \|w\|_2 \|e\|_{L^2} \leq \|w\|_{H^1} \|e\|_{L^2}. \)

By lemma 2 it follows that \( C\|w\|_{H^1}^2 \leq \|w\|_{H^1} \|e\|_{L^2}, \) so

\[ \|w\|_{H^1} \leq C\|e\|_{L^2}. \tag{9} \]

On the other hand,

\[
\int_D e^2 = -\int_D e\Delta w = \int_D \nabla e \cdot \nabla w - \int_{\partial D} e \frac{\partial w}{\partial n} = \int_D \nabla e \cdot \nabla (w - \pi_h w) - \int_{\partial D} \frac{\partial w}{\partial n}.
\]

We denote the piecewise linear interpolant of \( w \) by \( \pi_h w \). We have used equation (7) and we introduce \( \epsilon \in H^1(D) \), found in part (c) of the previous exercise, which satisfies \( \epsilon|_{\partial D} = (u - u_h)|_{\partial D} = g_D - g|_{\partial D}. \) It follows that

\[ \|e\|_{L^2}^2 \leq \|\epsilon\|_1 |w - \pi_h w|_1 + \|\epsilon\|_1 |w|_1 + \|\epsilon\|_{L^2} \|e\|_{L^2}. \tag{10} \]

Now we estimate the first two terms on the right of (10) by interpolation theory, theorem 1, elliptic regularity, and equation (9):

\[
\|\epsilon\|_1 |w - \pi_h w|_1 \leq C|\epsilon|_1 |Ch|w|_2 + |\epsilon|_1 \|w\|_{H^1} + \|\epsilon\|_{L^2} \|e\|_{L^2} \leq C|\epsilon|_1 |Ch|u_2 Ch \|e\|_{L^2} + |\epsilon|_1 \|w\|_{H^1} + \|\epsilon\|_{L^2} \|e\|_{L^2} \\
\leq C \epsilon_1 |u_2|_2 \|e\|_{L^2} + C|\epsilon|_1 \|e\|_{L^2} + \|\epsilon\|_{L^2} \|e\|_{L^2}.
\]

Thus \( \|u - u_h\|_{L^2} \leq C h^2 |u|_2 + C|\epsilon|_1 + \|\epsilon\|_{L^2} \). From the exercise,

\[
\|u - u_h\|_{L^2} \leq C h^2 |u|_2 + C h^{3/2} \|g_D''\|_\infty + C h^{5/2} \|g_D''\|_\infty \leq C_1 h^2 |u|_2 + C_2 h^{3/2} \|g_D''\|_\infty.
\]

This theorem substantially, but not completely, explains the convergence observed in example 1 and figure 1.

References