Additional XI. Suppose $f : [a, b] \to \mathbb{R}$. We define the lower envelope $g$ of $f$ to be the function defined by

$$g(y) = \sup_{\delta > 0} \inf_{|x-y| < \delta} f(x),$$

and the upper envelope $h$ by

$$h(y) = \inf_{\delta > 0} \sup_{|x-y| < \delta} f(x).$$

Show $g(x) \leq f(x) \leq h(x)$ for all $x$. Show that $f$ is continuous at $x$ if and only if $g(x) = h(x)$.

[Compare to 2.51 on page 52.]