Assignmen #3

(All Problems Due Monday 10/1/01.)

Section 2.4, # 21.
Section 2.5, # 24.
Section 2.5, # 31.
Section 4.1, # 1.
Section 3.2, # 5.

Additional V. Read section 2.5 on the topology of the real line. Does proposition 9 follow as an immediate corollary from propositions 7 and 8? (That is, did Royden mistakenly use Lindelöf’s proof when he could have just written ”...follows from propositions 7 and 8”?)

Additional VI. Let

\[ g(x) = \begin{cases} 
\frac{1}{m}, & x = \frac{a}{m} \text{ in lowest terms} \\
0, & x \text{ irrational} 
\end{cases} \]

Show \( g \) is Riemann integrable and that \( \int_{0}^{1} g(x) \, dx = 0. \)

Additional VII. (a) Prove that \( g \) (in the previous problem) is not continuous at \( x \) if and only if \( x \) is rational.

[Thus the set of discontinuities is of (Lebesgue and outer) measure zero. It is in fact a general truth (which you are not required to prove): a function \( f \) is Riemann integrable on a finite interval \([a, b]\) if and only if the set of discontinuities of \( f \) is of measure zero.]

(b) Prove that the function \( f \) defined by \( f(x) = 1 \) if \( x \) rational and \( f(x) = 0 \) if \( x \) irrational is discontinuous at every point of the interval \([0, 1]\).

Additional VIII. (Replaces 3.2 # 6.) Prove that given any set \( A \subset \mathbb{R} \) and any \( \epsilon > 0 \), there is an open set \( O \) such that \( A \subset O \) and \( M^*O \leq M^*A + \epsilon. \)

[Note: The problems from 2.4 and 2.5 are the last review problems!]