# On Legendre Polynomials

**Plot of Legendre polynomials.** I wrote this short program to plot some Legendre polynomials, and to show the roots of $P_4(x)$. It produced Figure 1.

*plotlegendre.m*

```
% PLOTLEGENDRE Plot five Legendre polynomials P_0(x),...,P_4(x), and
%   show the zeros of P_4(x). These points are x_1,x_2,x_3,x_4 in
%   the n=4 Gaussian quadrature ("Gauss-Legendre") rule.

x = -1:.002:1;
P2 = (1/3) * (3*x.^2 - 1);
P3 = (1/5) * (5*x.^3 - 3*x);
P4 = (1/35) * (35*x.^4 - 30*x.^2 + 3);
set(0,'defaultlinelinewidth',2.0)
plot(x,ones(size(x)),x,x,x,P2,x,P3,x,P4)
legend('P_0(x)','P_1(x)','P_2(x)','P_3(x)','P_4(x)',...
       'location','southeast')
grid on,  xlabel x, axis([-1 1 -1.2 1.2])

z4 = [0.3399810436 0.8611363116]; % from text: zeros of P_4
z4 = [-fliplr(z4) z4]             %   ... shows them as numbers
hold on
plot(z4,zeros(1,4),'o','markersize',12,'linewidth',3.0),  hold off
```

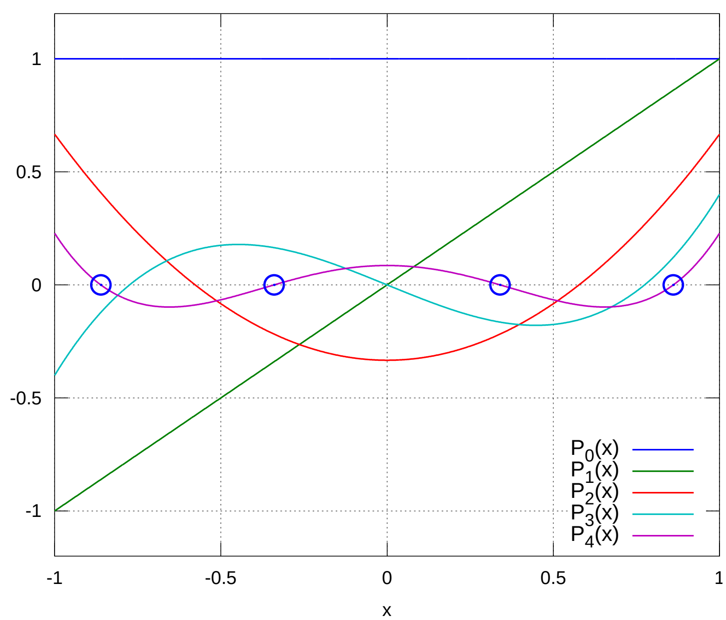


FIGURE 1. Plot of Legendre polynomials $P_0(x), \ldots, P_4(x)$. Result of `plotlegendre.m`.

**Definition.** Recall the textbook's definition of the Legendre polynomials:

(1a) for $n = 0, 1, 2, \ldots$, $P_n(x)$ is a polynomial of degree $n$,

(1a) $P_n(x)$ is *monic*, meaning it has leading coefficient 1, so $P_n(x) = x^n$+(lower order terms), and

(2) for every $P(x)$ which has degree less than $n$,

$$\int_{-1}^{1} P(x)P_n(x)\,dx = 0.$$

It is not immediately clear that this is possible, that is, that such polynomials exist! But they can be *constructed* by the method of "orthogonalizing" the monomials $\{1, x, x^2, x^3, \ldots\}$. If you have taken linear algebra then you should have seen the "Gram-Schmidt" process, and this can be applied to polynomials using the idea that the integral above is like a dot product,

$$\text{“}P \cdot Q\text{”} = \int_{-1}^{1} P(x)Q(x)\,dx.$$

It can also be shown that there is only one set of polynomials that satisfies the above definition.

**Example of Gauss' division step.** Calculation of quotient and remainder by long division.

*Given*: $n = 4$, $P_4(x) = \frac{1}{35}(35x^4 - 30x^2 + 3) = x^4 + 0x^3 - \frac{6}{7}x^2 + 0x + \frac{3}{35}$, and a polynomial of degree $2n - 1 = 7$:

$$P(x) = x^7 + x^6 + x^5 + 2x^4 - 2x^3 + x^2 + x + 4.$$

*Goal*: Compute $Q, R$ so that $P(x) = Q(x)P_4(x) + R(x)$.

*Solution.*