Solutions to Practice Final.

Recent Stuff:

1. [This question I would not ask in Fall 2000:] Get

$$y = a_0\left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots\right) = a_0 e^{-x}.$$

2. $\int_0^\infty e^{-st} t\,dt = \dots$ (integration–by–parts) $\dots = \frac{1}{s^2}$.

3.

$$Y(s) = \frac{e^{-3s}(1+3s)}{s^2(s^2+4)} + \frac{1}{s^2+4}.$$

4. [Ignore first sentence. I would not ask the question this way in Fall 2000.] Get $y(0) = a_0$, $y'(0) = 1$ and $y''(0) = -2a_0$, so $y(x) \approx a_0 + x - a_0 x^2$.

5. A. One point: $(0,0)$.
B. The phase plane equation is $\frac{dy}{dx} = \frac{x}{2y}$. This is separable and gives $y^2 = \frac{1}{2}x^2 + C$, which is a hyperbola.

6. Easy. Note $Y(s) = \frac{s+1}{(s+7)(s+1)} = \frac{1}{s+7}$ by cancellation. Thus $y(t) = e^{-7t}$.

7. By taking Laplace transforms, get

$$Y(s) = \frac{1}{s^2+2s-15}G(s) + 8\frac{1}{s^2+2s-15}.$$

The denominators factor, and we find $\frac{1}{s^2+2s-15} = \frac{1}{8}\left(\frac{-1}{s+5} + \frac{1}{s-3}\right)$. Thus

$$\begin{aligned} y(t) &= \frac{1}{8}\left(-e^{-5t} + e^{3t}\right) * g(t) + \left(-e^{-5t} + e^{3t}\right) \\ &= \frac{1}{8}\int_0^t \left(-e^{-5(t-v)} + e^{3(t-v)}\right) g(v)\,dv - e^{-5t} + e^{3t}. \end{aligned}$$

Comprehensive Stuff:

1. This calls only for implicitly differentiating: $2x + 2y\frac{dy}{dx} = 0$ thus $\frac{dy}{dx} = -\frac{x}{y}$, which is as claimed.

2. The characteristic equation is $r^2 + 6r + 11 = 0$. It has solutions $r = \frac{-6\pm\sqrt{36-44}}{2} = -3 \pm i\sqrt{2}$. Thus the general solution is

$$y(x) = c_1 e^{-3x}\cos\sqrt{2}x + c_2 e^{-3x}\sin\sqrt{2}x.$$

3. This is separable and equivalent to: $y^2 dy = x \sin x\, dx$. So

$$\frac{y^3}{3} = \int x \sin x\, dx = -x \cos x + \sin x + C.$$

Find $C = \frac{1}{3} - \pi$. One way to write the answer: $y(x) = \sqrt[3]{3\left(-x\cos x + \sin x + \frac{1}{3} - \pi\right)}$.

4. A. $\frac{dy}{dt} = K(M - y)$.
B. Its separable. Solve to get: $-\ln|70 - y| = \frac{1}{30}t + C$. Find $C = -\ln 50$, and then $y(t) = 70 + 50e^{-\frac{1}{30}t}$. And $y(60) = 70 + 50e^{-2}$.

5. A. First order linear. Get $w(x) = \frac{3}{5}x^2 - \frac{1}{2}x + Cx^{-3}$.
B. "Homogeneous": $\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x}\right) - \frac{3}{2}\frac{1}{(y/x)}$. Substitute $xv = y$ and $v + xv' = y'$ to get a separable equation for v with solution $v^2 + 3 = A|x|^{-1}$ or $y = \pm x\sqrt{A|x|^{-1} - 3}$.

6. A. 1. Check that they are solutions by substitution into the equation $y'' - 4y = 0$. **2.** And $W[y_1, y_2] = -4$, so they are lin. independent.
B. Substitution.
C. Since $y(x) = c_1 e^{2t} + c_2 e^{-2t} - \frac{1}{5}\sin t - \frac{1}{4}t^2 - \frac{1}{8}$ is the general solution to the nonhomogeneous equation, we just use the initial conditions to find $c_1 = \frac{1}{20}$ and $c_2 = -\frac{1}{20}$.