Assignment # 6
Due Wednesday 11/5 at start of class

Exercise 10.1.

Exercise 11.2. (a)

Exercise 11.3.

IX. What good are the orthogonal functions \( q_j(x), j = 1, \ldots, 5 \), which were the columns of \( Q \) in problem VIII? One answer is that they can sometimes help to approximately solve hard problems which relate in some manner to the functions \( e^{-jx} \). For instance:

\[ \text{Problem. Find a smooth function } \hat{u}(x), \text{ defined for } x \in [0, \infty), \text{ which approximately solves} \]
\[ -u''(x) + (5 + 4 \cos(x)) u(x) = 0, \quad u(0) = 1, \quad \lim_{x \to +\infty} u(x) = 0. \]

Of course, the sense of “approximation” is vague, and I will leave it so. (Try to come up with a better approximation than the one given, using only four coefficients! By the way, I do not know how to solve this problem by hand. Do you?)

(a) Argue informally that the exact \( u(x) \) satisfies \( e^{-3x} \leq u(x) \leq e^{-x} \). (Hint: Compare the problem to one with “5 + 4 cos(x)” replaced by constants. Solve those problems exactly.)

Now consider (and run!) the following MATLAB:

\[
\begin{align*}
&>> B=\text{hilb}(6); \quad B=B(1:5,2:6); \quad R=\text{ chol}(B); \\
&>> x=(0:.01:3)'; \quad A=\left[ \exp(-x) \exp(-2*x) \exp(-3*x) \exp(-4*x) \exp(-5*x) \right]; \\
&>> Q=A/R; \quad \text{plot}(x,Q(:,5)), \text{ grid on} \\
&>> c=\text{roots}(\text{flipud}(R\backslash([\text{zeros}(1,4) \ 1]'))); \quad z=-\text{log}(c); \quad z' \\
&\text{ans} = \\
&\quad 0.0588 \quad 0.3241 \quad 0.8761 \quad 1.9678
\end{align*}
\]

(b) Explain why \( z \) contains the four finite solutions of \( q_5(x) = 0 \).

Suppose \( \hat{u}(x) \) is a linear combination of \( q_1(x), \ldots, q_4(x) \): \( \hat{u}(x) = \sum_{j=1}^{4} c_j q_j(x) \). A **spectral collocation method** for solving the problem above is to use the values \( z_1, \ldots, z_4 \), and the boundary condition at \( x = 0 \), to determine \( c_j \) (and thus \( \hat{u}(x) \)) by the following prescription:

\[
\begin{align*}
(1) & \quad \sum_{j=1}^{4} c_j q_j(0) = 1, \\
(2) & \quad \sum_{j=1}^{4} c_j \left( -q_j''(z_k) + (5 + 4 \cos(z_k)) q_j(z_k) \right) = 0, \quad k = 1, 2, 3, 4.
\end{align*}
\]
This is five equations, the first of which is the boundary condition, and the next four are requiring the differential equation to be true at the roots of \( q_5(x) \). As there are only four unknown coefficients, the system is overdetermined.

\((c)\) Using either a least squares method to approximately solve equations (1) and (2), or removing the \( z_4 \) equation from equations (2) and solving the resulting system, find \( c_j \) and plot the solution \( \hat{u}(x) \).

\textit{Extra Credit:}

\textbf{X (Extra Credit).} Now find some other way to approximately solve the above boundary value problem, presumably using a method (finite differences?) with lots of degrees of freedom, and compare. Explain what you do with the boundary at infinity in this context.