Assignment #9 = Take-home Final Exam

Due Wednesday 12 May, 2010 at 5pm in my office or mailbox.

Rules. You may not talk or communicate about this exam with any person other than me (Ed Bueler): elbueler@alaska.edu, x7693. You may use any reference, print or electronic, as long as it is clearly cited, but you may not search out complete solutions to these particular problems, whether or not they exist. Please refer specifically to ideas and/or equations in the textbook if that is needed for clarity. This assignment is worth a total of 60 points, twice the usual value.

0. Read sections 4.9, 6.1, 6.2, and 6.3 of the textbook MORTON & MAYSERS, 2ND ED. This assignment will also cover earlier sections of the textbook, of course.

1. (10 points) In section 2.15 an explicit, centered-difference $O(\Delta t, \Delta x^2)$ method is considered for problem like this one:

$$u_t^* = b(x,t)u_{xx} - a(x,t)u_x.$$

Recall that the analysis of this explicit method shows that a maximum principle holds, and thus convergence and stability occur, if there are two conditions, (2.144) and (2.145). A natural question: Can implicitness remove both conditions and give unconditional convergence?

To answer this question, consider the simplest implicit centered-difference scheme for equation *, the scheme which only involves grid values $U_j^n, U_j^{n+1}, U_{j+1}^{n+1}$. State the scheme and draw its stencil. Show that this scheme converges only conditionally, and that in fact the restriction (2.146) on mesh Péclet number remains.

(There is no need to implement the scheme. Also, a hint: Section 2.11 and my in-class lecture on 15 March each illustrate how to do a maximum principle argument for an implicit scheme.)

2. (a) (6 points) Implement in MOP the explicit method (2.153) for equation (2.150). This is a “staggered grid” method in that the diffusivity $p(x,t)$ is evaluated half-way between regular grid points $\{x_j\}$. Test the convergence of this scheme on this PDE with an added source term,

$$u_t = \left(e^{2(x-1)}u_x\right)_x + f(x,t).$$

Use Dirichlet boundary conditions $u(0,t) = 0, u(1,t) = 0$ on the interval $0 \leq x \leq 1$. Choose an initial condition $u(x,0)$ and a right-hand side function $f(x,t)$ so that you know the exact solution $u(x,t)$. Test the program by showing clearly that the maximum error at $t = 0.2$ decreases over these J-values: $J = 20, 40, 80$.

(b) (6 points) Now implement the centered explicit scheme (2.141) for the same problem, but written in generic conduction-advection form,

$$u_t = p(x,t)u_{xx} + p_x(x,t)u_x + f(x,t).$$

This second scheme uses only “regular” and not “staggered” grid values of $p(x,t)$. Test this second implementation on the same problem as in part (a). By comparing results from parts (a) and (b), do you have evidence for the claim “but it is usually better” near the top of page 51? Explain in a few sentences.
3. (a) (4 points) Implement the scheme from 2(a) on the $n = 1$ porous medium equation, a nonlinear “heat-like” equation,

$$u_t = (u u_x)_x,$$

wherein $p(x, t) = u(x, t)$ is the solution itself. In particular, choose a reasonable way for approximating the staggered grid values $p_{j+1/2}$ from the regular-grid values of $u$.

(b) (4 points) Show that this formula

$$u(x, t) = \begin{cases} 
6^{-1}(t + 1)^{-1/3} \left(1 - x^2(t + 1)^{-2/3}\right), & |x| < (t + 1)^{1/3}, \\
0, & |x| \geq (t + 1)^{1/3},
\end{cases}$$

gives a (not-at-all-obvious!) exact solution. In showing this, consider only the set where $|x| < (t + 1)^{1/3}$ because the other sets are either subtle or trivial.

(c) (4 points) Test your code from part (a) on this exact solution, on the interval $|x| \leq 2$, taking your initial values $u(x, 0)$ from the values of the exact solution. Evaluate the numerical error at time $t = 1.0$ and use $J = 40, 80$.

4. (10 points) Reproduce figure 4.15, by the method described in the textbook but with one modification: use one step of FTCS (forward-time-centered-space) to get leap-frog started. 2

5. (10 points) Implement method (6.29b) for equation (6.20) in MORTON & MAYERS, 2ND ED. In particular, approximate the solution $u(x, y)$ on the square $(x, y) \in [-1, 1] \times [-1, 1]$ if

$$a(x, y) = 1.3 + \cos(\pi y)$$

and

$$f(x, y) = \begin{cases} 
1, & (x - 0.1)^2 + (y + 0.4)^2 < 0.16, \\
0, & \text{otherwise}.
\end{cases}$$

Use boundary conditions $u = 0$ on all edges of the square. Be sure to use sparse matrix storage. Solve the linear system by “A\b”. Show the result for the $\Delta x = \Delta y = 0.1$ and $\Delta x = \Delta y = 0.01$ cases. State the sizes of these two matrix problems and time them using tic, toc or equivalent.

6. (6 points) Consider again the linear advection equation $u_t + au_x = 0$ with $a \geq 0$ constant. Let $\nu = a\Delta t/\Delta x$. For the implicit upwind method

$$U_{j+1}^n - U_j^n + \nu(U_{j+1}^{n+1} - U_{j-1}^{n+1}) = 0,$$

(i) compute the truncation error, and
(ii) show by a maximum principle argument, starting as usual with the definition $e_j^n = U_j^n - u(x_j, t_n)$, that the scheme is unconditionally convergent.

Thus there exists at least one scheme that does not require CFL for convergence, but it is implicit. Section 4.8 illustrates another implicit scheme for advection equations, but of higher accuracy.

2The left column of the figure shows a solution of the problem stated on page 98 of the textbook, while the right column shows the problem with initial condition stated on page 103. It may be useful to see the online examples http://www.dms.uaf.edu/~bueler/upwindfigure.m and/or http://www.dms.uaf.edu/~bueler/lwfigure.m.
3Here is a physical interpretation of this PDE problem: $u(x, y)$ is the equilibrium (steady) temperature of a square plate with nonconstant conductivity $a(x, y)$ and heat source distribution $f(x, y)$. 