Assignment #8
Due Monday April 26, 2010.
MOP = MATLAB, OCTAVE, or PYLAB

0. Read sections 4.1, 4.2, 4.3, 4.4, 4.5.

1. (a) For the problem

\[ u_t + (1 - t^2)u_x = 0, \quad u(x,0) = \arctan x, \]

and for \( x \in \mathbb{R} \) and \( t \geq 0 \), sketch the characteristics in the \((x,t)\) plane. In particular, sketch those characteristics which go through (i.e. “start at”) the points with \( t = 0 \) and \(-3 \leq x \leq 3\). (Hints for a good sketch: What are the initial slopes? At what time does the advection change direction?)

(b) Solve the problem by hand. That is, apply the method of characteristics. Check your answer \( u(x,t) \) by substitution into the PDE. (Your answer should be correct for the half-plane \( t \geq 0 \).)

(c) Write a MOP program to just plot the exact solution \( u = u(x,t) \). In particular, plot this solution as a mesh on the rectangle \(-3 \leq x \leq 3, 0 \leq t \leq 2\), and as a contour plot on the same rectangle. (The contours will be familiar curves . . .)

2. (a) Solve by hand for \( x \in \mathbb{R} \) and \( t \geq 0\):

\[ u_t + (x t^2)u_x = 0, \quad u(x,0) = \sin x. \]

(b) Solve by hand for \( x \in \mathbb{R} \) and \( t \geq 0\):

\[ u_t + (x t^2)u_x = 1, \quad u(x,0) = \sin x. \]

(Hint: Recall what property \( u(x(t),t) \) has, that is, how \( u(x,t) \) behaves along a characteristic. There is no need to start over on what you did in part a. Simply address where to add a bit of new stuff, roughly at the end of the argument for part a, and then easily solve this new problem. Then check your answer.)

3. Exercise 4.1, page 146. (You do not have to write any code for this one. But it may be helpful to your understanding.)

4. (a) The Lax-Wendroff method for \( u_t + a(x,t)u_x = 0 \) is stated as equation (4.44) in the text. The truncation error of (4.44) is \( O(\Delta t, \Delta x^2) \) if the solution is smooth enough. Generalize (4.44) slightly further to the case \( u_t + a(x,t)u_x = g(x) \), for given \( g(x) \), preserving the same order of accuracy. Carefully state the Lax-Wendroff scheme at this generality, in a form similar to (4.44).

(b) Write a MATLAB program which numerically approximates \( u(x,t) \) solving

\[ u_t + (x t^2)u_x = 1, \quad u(x,0) = \sin x, \quad u(0,t) = t, \]

for \( 0 \leq x \leq 3 \) and \( 0 \leq t \leq 2 \), using your Lax-Wendroff method from part a. Using \( \Delta x = 0.1 \), show the approximate solution \( u(x,t) \) at \( t = 2 \). Note that the CFL condition determines the time step \( \Delta t \) from the given values for \( \Delta x \), so you may do this adaptively or you may pre-compute fixed time step that satisfies CFL.

(c) Verify your solution at \( t = 2 \) using the exact solution in 2b. In particular, find the rate of convergence of the worst case error at \( t = 2 \) for these grid intervals: \( \Delta x = 0.2, 0.1, 0.05, 0.02, 0.01, 0.005 \). Estimate the rate of convergence as a power of \( \Delta x \).