Assignment #3

Due Wednesday, 24 January 2010 at start of class.

These problems merely duplicate the last three slides of the online lectures, 
http://www.dms.uaf.edu/~bueler/twopoint.pdf

1. Solve by-hand this ODE BVP to find \( y(x) \):

\[
y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y(1) = 0.
\]

2. Recall Example 3 in the slides, an impossible-to-solve ODE BVP. Nonetheless there are some values of \( A \) in the following problem which allow a solution: find \( y(x) \) if

\[
y'' + \pi^2 y = 0, \quad y(0) = 1, \quad y(1) = A.
\]

What values of \( A \) are allowed? For an allowed value of \( A \), how many solutions are there?

3. Equation (6) has non-constant coefficients, and essentially it cannot be solved exactly by-hand. To develop some sense of the effect of the source term \( s(x) \), solve by-hand this ODE BVP

\[
(k_0 u')' = -s(x), \quad u'(0) = 0, \quad u(L) = 0,
\]

merely assuming the source is quadratic \( s(x) = ax^2 + bx + c \) and the conductivity is constant \( k_0 > 0 \). Compute by-hand \( u(0) \). How does the solution \( u(x) \) depend on \( s(x) \)? (For example, how does \( u \) depend on the sign, values, slope, or concavity of \( s(x) \)?)

4. Apply the finite difference method to solve this ODE BVP:

\[
y'' + \sin(5x)y = x^3 - x, \quad y(0) = 0, \quad y(1) = 0.
\]

In particular, use \( J = 10 \), \( \Delta x = 1/J \), and \( x_j = j\Delta x \) for \( j = 0, \ldots, J \). Construct the system

\[
Ay = b
\]

where \( A \) is a \((J+1) \times (J+1)\) matrix, \( y = \{Y_j\} \) approximates the unknowns \( \{y(x_j)\} \), and \( b \) contains the right-side function \( x^3 - x \) in the ODE. Arrange things so that the first equation in the system represents the boundary condition \( y(0) = 0 \) and the last equation the condition \( y(1) = 0 \). The remaining equations in the system will each hold finite difference approximations of the ODE. Show me your matrix \( A \) in a non-wasteful way. Solve the system to find \( y \), and plot it appropriately. Also write a few sentences addressing how to know qualitatively and quantitatively the degree to which your answer is a good approximation.
5. (The goal of this problem is to understand shooting, though you will not quite put all parts together . . .)
Consider the nonlinear ODE BVP
\[ u'' + u^3 = 0, \quad u(0) = 1, \quad u(1) = 0. \]
This problem is well-suited to the shooting method. Specifically, write a MOP program that uses an ODE solver to solve the following ODE IVP
\[ u'' + u^3 = 0, \quad u(0) = 1, \quad u'(0) = A \]
for each of the eleven values \( A = -5, -4, \ldots, 4, 5 \). Plot all eleven solutions, and identify on the plot the \( A \) value for each curve. Which two \( A \) values make the computed value \( u(1) \) bracket the desired value (boundary condition) “\( u(1) = 0 \)”?
(With this information in hand you could make a program like varheatSHOOT.m, which uses bisection to converge to an \( A \) value so that \( u(1) \approx 0 \) to many-digit-accuracy.)

\[ ^1 \text{Possibly using the text command in MATLAB/Octave.} \]