Exercise 15.16. The equation given simplifies to

\[(c - d)u_{n+1} + 2d u_n - (c + d)u_{n-1} = 0\]

where \(c, d > 0\). We will need to assume \(c \neq d\) in what follows. (The actual situation when numerically-solving a differential equation in a reliable way would be \(c \ll d\). We are asked, also, to show what happens when \(c > d\).) The characteristic equation for this recursion relation, corresponding to \(u_n = \lambda^n\), is

\[c - d)\lambda^2 + 2d\lambda - (c + d) = 0\]

This equation has roots

\[\lambda = \frac{-2d \pm \sqrt{4d^2 + 4(c - d)(c + d)}}{2(c - d)} = \left\{1, \frac{d + c}{d - c}\right\}.

(It is not surprising that one root is 1, in retrospect, because the sum of all coefficients in the original recursion is zero.)

The general solution to the recursion is

\[u_n = c_1 + c_2((d + c)/(d - c))^n\]

The boundary conditions show \(c_1 + c_2 = 0\) and \(c_1 + c_2 ((d + c)/(d - c))^M = 1\), and we get after a calculation

\[u_n = \frac{-(d - c)^M((d + c)/(d - c))^n}{(d + c)^M - (d - c)^M} = \frac{-(d - c)^M(d - c)^n + (d + c)^M(d + c)^n}{(d - c)^n[(d + c)^M - (d - c)^M]}\]

it is easy to check that this satisfies the boundary conditions and the recursion.

We want to consider the ratio of successive values, and show that if \(c > d\) then this ratio is negative (so successive values have opposite signs). But

\[\frac{u_{n+1}}{u_n} = \frac{-(d - c)^M + (d - c)^M((d + c)/(d - c))^{n+1}}{-(d - c)^M + (d - c)^M((d + c)/(d - c))^n} = \frac{(d + c)/(d - c))^{n+1} - 1}{(d + c)/(d - c))^n - 1}.

Now, if \(c, d > 0\) and \(c > d\) then \((d + c)/(d - c) < -1\). If \(n\) is even, for example, it then follows that the denominator of the above fraction is positive while the numerator is negative. The opposite holds if \(n\) is odd. Thus the ratio is odd.

A context for this problem: Consider the boundary value problem for the equilibrium temperature distribution \(u = u(x)\) in a material with conductivity \(K > 0\). Suppose the material is also moving uniformly with constant velocity \(v\) and that the boundary temperatures are specified. The problem is then

\[\frac{\partial u}{\partial x} = K \frac{\partial^2 u}{\partial x^2}, \quad u(0) = 0, \quad u(1) = 1.

This is called an conduction/advection problem. Suppose one approximates by finite differences, letting \(\Delta x = 1/M\) and \(x_n = n\Delta x\). Suppose \(u_n \approx u(x_n)\). Then \(u_0 = 0\) and \(u_M = 1\) by the boundary conditions. We replace

\[\frac{\partial u}{\partial x} \rightarrow \frac{u_{n+1} - u_{n-1}}{2\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2},\]

these are the standard centered-difference approximations. The differential equation is replaced by

\[\frac{v}{2\Delta x} = K \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} = \frac{v}{2\Delta x} = K \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2}.

With the identifications \(c = v/(2\Delta x)\) and \(d = K/(\Delta x^2)\), this is exactly the recursion equation which is the subject of the exercise. Note \(\Delta x\) is presumably small, and, in particular, as \(\Delta x \to 0\) it is eventually the case that \(c \ll d\) as long as \(v > 0\) and \(K > 0\).
The solution of the differential equation can actually be found by hand. (Thus this is a textbook example and not a full “real-world” problem.) On the other hand, how accurate is the finite difference solution \( u_n \)? Clearly the answer depends on \( M \), and, in theory, the exact solution is recovered in the limit \( M \to \infty \). But, in fact, the last comment in the original exercise gives a minimum \( M \) for which the approximate solution \( u_n \) is even qualitatively right. That is, if \( c > d \) then there is a qualitatively wrong fact about \( u_n \); it oscillates.

In the context of numerical solution of differential equations, we have to interpret this oscillation as instability. See figure 1. Note that when \( M = 6 \) we have \( c > d \), and the solution \( u_n \) is clearly wrong, while for \( M = 16 \) we have \( c < d \) and there is a reasonable solution (though still not the exact solution).

\[ y(x) = k y_1(x) + c y_2(x) \quad \text{where} \quad y_1(x) = \frac{1}{(x - 2)^2} \left( \frac{2}{3} \left( x - \frac{1}{2} \right) - \frac{1}{2} \right), \quad y_2(x) = \frac{x^2}{(x - 2)^2}. \]

Substitution of \( y_1 \) into the differential equation is a completely tedious exercise, but is simply calculus. Similarly for the substitution of \( y_2 \).

What remains after such substitutions is to show that the two solutions are linearly-independent. This case, in which one has acquired the solutions by unspecified means, is a case in which the

![](image.png)
Wronskian is a useful concept. Indeed, the Wronskian
\[ W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1 \]
is nonzero if and only if \( y_1 \) and \( y_2 \) are linearly-independent. Computing the Wronskian is merely tedious.

In this case we can see that the solutions are linearly-independent more directly. Namely, \( y_1 \) and \( y_2 \) have the same denominator. Thus linear-dependence would only be true if there were nonzero constants \( a, b \) so that
\[ a \left( \frac{2}{3x} - \frac{1}{2} \right) + bx^2 = 0 \]
for all \( x \neq 2 \). If \( x = 0 \) this relation shows \( a = 0 \), while if \( x = 4/3 \) this relation shows \( b = 0 \), so linear-dependence is impossible.

**Exercise 15.24b.** Here we do the whole story: First we find the general solution to the homogeneous problem:
\[ y_h = c_1 e^x + c_2 xe^x. \]
This follows from the characteristic polynomial \( \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \) so \( y_1 = e^x \) is one solution and, because of the repeated root, \( y_2 = xe^x \) is the another linearly-independent solution.

Now we seek a particular solution of the form
\[ y_p = k_1(x)y_1(x) + k_2(x)y_2(x), \]
that is, by variation of parameters, and we get a system of equations for \( k'_1, k'_2 \):
\[ k'_1 e^x + k'_2 xe^x = 0, \]
\[ k'_1 e^x + k'_2 (1 + x)e^x = 2xe^x. \]
[\text{I find it worthwhile to think through the derivation of this result!}]
It is easy to solve, for instance by subtracting the equations, to get \( k'_2 = 2x \) so \( k_2(x) = x^2 \). (Note we need not keep any constants of integration at this stage because that will merely repeat a “piece of” a homogeneous solution.) Then \( k'_1 = -xk'_2 = -2x^2 \) so \( k_1(x) = -(2/3)x^3 \). Thus
\[ y_p(x) = -\frac{2}{3}x^3 e^x + x^2 xe^x = \frac{1}{3} x^3 e^x. \]
This is straightforward to check (with modest product rule skills!). The general solution is
\[ y(x) = c_1 e^x + c_2 xe^x + \frac{1}{3} x^3 e^x. \]
Needless to say, this result can also be achieved by the method of undetermined coefficients.

**Exercise 15.27.** This problem asks us to do the Green’s function calculation again, but this time in the abstract. It turns out not to be harder than previous concrete examples.

We are told to assume that we have linearly-independent solutions \( y_1 \) and \( y_2 \). At the appropriate time we will use the facts \( y_1(0) = 0 \) and \( y_2(1) = 0 \). For now, let’s write down \( G(x, \xi) \) for the problem consisting of the given ODE and the boundary values \( y(0) = 0 \) and \( y(1) = 0 \):
\[ G(x, \xi) = \begin{cases} 
  c_1 y_1(x) + c_2 y_2(x), & 0 < x < \xi, \\
  d_1 y_1(x) + d_2 y_2(x), & \xi < x < 1.
\end{cases} \]
We have introduced four unknown constants and thus need four conditions. First, the boundary conditions and the given facts about \( y_1, y_2 \) imply

\[
0 = c_1 \cdot 0 + c_2 y_2(0) \quad \text{and} \quad 0 = d_1 y_1(1) + c_2 \cdot 0.
\]

Assuming, generically, that \( y_2(0) \) and \( y_1(1) \) are not zero—in fact we can prove they are not; how?—we get \( c_2 = 0 \) and \( d_1 = 0 \) so

\[
G(x, \xi) = \begin{cases} 
  c_1 y_1(x), & 0 < x < \xi, \\
  d_2 y_2(x), & \xi < x < 1.
\end{cases}
\]

Next, continuity of \( G(x, \xi) \) at \( x = \xi \) implies \( c_1 y_1(\xi) = d_2 y_2(\xi) \). The usual jump condition on the first derivative of \( G \)—it’s worth rederiving!—says

\[
\frac{\partial G}{\partial x}(\xi^+, \xi) - \frac{\partial G}{\partial x}(\xi^-, \xi) = d_2 y_2(\xi) - c_1 y_1(\xi) = 1.
\]

Rewriting the last two equations, we have a system of two equations in two unknowns for \( c_1, d_2 \):

\[
\begin{align*}
y_1(\xi)c_1 - y_2(\xi)d_2 &= 0, \\
y_1(\xi)c_1 + y_2(\xi)d_2 &= 1.
\end{align*}
\]

Combining these equations leads us to the Wronskian whether we like it or not:

\[
d_2 = y_1(\xi)/W(\xi), \quad c_1 = y_2(\xi)/W(\xi),
\]

where \( W(\xi) = y_1(\xi)y_2(\xi) - y_1(\xi)y_2(\xi) \). This is the desired result.

**Exercise 15.31.** The solution to the homogeneous equation \( \ddot{x} + \alpha \dot{x} = 0 \) is \( x_h(t) = c_1 + c_2 e^{-\alpha t} \).

Thus the Green’s function which solves \( \ddot{x} + \alpha \dot{x} = \delta(t - t_0) \), for \( t_0 > 0 \) and \( t > 0 \), is

\[
G(t, t_0) = \begin{cases} 
  c_1 + c_2 e^{-\alpha t}, & 0 \leq t < t_0, \\
  d_1 + d_2 e^{-\alpha t}, & t_0 < t < \infty.
\end{cases}
\]

The initial conditions only apply to the first case of the formula for the Green’s function: \( c_1 + c_2 = 0 \) and \( -\alpha c_2 = 0 \) imply \( c_1 = c_2 = 0 \). The continuity of \( G(t, t_0) \) at \( t = t_0 \) implies \( 0 = d_1 + d_2 e^{-\alpha t} \). The jump condition

\[
\frac{\partial G}{\partial t}(t_0^+, t_0) - \frac{\partial G}{\partial t}(t_0^-, t_0) = 1
\]

becomes \( -\alpha d_2 e^{-\alpha t_0} = 1 \). We conclude

\[
G(t, t_0) = \begin{cases} 
  0, & 0 \leq t < t_0, \\
  \alpha^{-1}(1 - e^{\alpha(t_0 - t)}), & t_0 < t < \infty,
\end{cases}
\]

and that

\[
x(t) = \int_0^\infty G(t, t_0) f(t_0) dt_0 = \int_0^t \alpha^{-1} \left(1 - e^{\alpha(t_0 - t)}\right) f(t_0) dt_0
\]

is the solution to the general nonhomogeneous equation \( \ddot{x} + \alpha \dot{x} = f(t) \).

Now, when \( f(t) = A e^{-\alpha t} \) we can do the integral:

\[
x(t) = \int_0^t \alpha^{-1} \left(1 - e^{\alpha(t_0 - t)}\right) A e^{-\alpha t_0} dt_0 = A \alpha^{-1} \int_0^t e^{-\alpha t_0} - e^{-\alpha t} e^{\alpha t_0 (a - \alpha)} dt_0
\]

\[
= A \alpha^{-1} \left[ a^{-1}(1 - e^{-\alpha t}) - e^{-\alpha t}(a - \alpha)^{-1}(1 - e^{\alpha(a - \alpha)}) \right]
\]

The last two forms are not obviously equivalent, but after some work I saw that the form I computed (the first) matches the solution in the text.