Selected Solutions to Assignment #3

Exercise 5 (page 32 of B&C). Let \( S \) be the open set of all points such that \(|z| < 1\) or \(|z-2| < 1\).

State why \( S \) is not connected.

**Proof.** There are no polygonal line that connects points \((0,0)\) and \((2,0)\). Such a line necessarily crosses the line \(\{x = 1\}\), which does not belong to \( S \). (It is helpful to draw the picture.) □

Exercise 6 (page 32 of B&C). Show that a set \( S \) is open if and only if each point in \( S \) is an interior point.

**Proof.** (This proof follows the book’s definition. It is also possible to use the definition given in class, in which case the proof is just as short.)

Let \( S \) be open. Then by definition, it does not contain boundary points, so all points in \( S \) are interior. Let now \( S \) be a set consisting of interior points, then by definition, it is open (it does not contain boundary points). □

Exercise 9 (page 32 of B&C). Show that any point \( z_0 \) of a domain \( S \) is an accumulation point of that domain.

**Proof.** Consider some deleted neighborhood of \( z_0 \): \( D(z_0) = \{0 < |z - z_0| < \varepsilon\} \). This deleted neighborhood is not necessarily contained in \( S \). Since \( S \) is a domain, it is open, so there exist some \( \delta > 0 \) such that neighborhood \( B(z_0) = \{0 < |z - z_0| < \delta\} \) is contained in \( S \). Then the set \( D(z_0) \cap B(z_0) \) is not empty, so the deleted neighborhood of \( D(z_0) \) contains points from \( S \). Since \( \varepsilon \) was arbitrary, we have proven that any deleted \( \varepsilon \) neighborhood of \( z_0 \) contains points of \( S \), so \( z_0 \) is an accumulation point of \( S \). □

Exercise C3. Write

\[
f(z) = \frac{1}{1+z} + \sqrt{z}
\]

in the form \( f(z) = u(r, \theta) + iv(r, \theta) \)

**Solution.** Note \( z = r(\cos \theta + i \sin \theta) \). and \( \sqrt{z} = \sqrt{r}(\cos \theta/2 + i \sin \theta/2) \). We arrive at

\[
\frac{1}{1+z} + \sqrt{z} = \frac{1}{(r \cos \theta + 1) + ir \sin \theta} + \sqrt{r}(\cos \theta/2 + i \sin \theta/2)
\]

We multiply the numerator and denominator of fraction by \( r \cos \theta + 1 - ir \sin \theta \) and have that

\[
\frac{1}{1+z} + \sqrt{z} = \frac{r \cos \theta + 1 - ir \sin \theta}{(r \cos \theta + 1)^2 + (r \sin \theta)^2} + \sqrt{r}(\cos \theta/2 + ir \sin \theta/2) =
\]

\[
\left( \frac{r \cos \theta + 1}{r^2 + 1 + 2r \cos \theta} + \sqrt{r} \cos \theta/2 \right) + i \left( \frac{-r \sin \theta}{r^2 + 1 + 2r \cos \theta} + \sqrt{r} \sin \theta/2 \right)
\]

Thus

\[
u(r, \theta) = \frac{r \cos \theta + 1}{r^2 + 1 + 2r \cos \theta} + \sqrt{r} \cos \theta/2 \quad \text{and} \quad v(r, \theta) = \frac{-r \sin \theta}{r^2 + 1 + 2r \cos \theta} + \sqrt{r} \sin \theta/2.
\]
Exercise 3 (page 42 of B&C). Sketch the region onto which the sector \( r \leq 1, 0 \leq \theta \leq \pi/4 \) is mapped by the transformation

a) \( \omega = z^2 \): Solution. Upper right quarter of circle:
\[ D = \{ r \leq 1, 0 \leq \theta \leq \pi/2 \} \]

b) \( \omega = z^3 \): Solution.
\[ D = \{ r \leq 1, 0 \leq \theta \leq 3\pi/4 \} \]

c) \( \omega = z^4 \): Solution. Upper half of circle:
\[ D = \{ r \leq 1, 0 \leq \theta \leq \pi \} \]

Exercise 5 (page 42 of B&C). Verify that the image of the region \( a \leq x \leq b, c \leq y \leq d \) under the transformation \( \omega = e^z \) is the region \( e^a \leq \rho \leq e^b, c \leq \phi \leq d \).

Proof. If \( z = x + iy \) then
\[ w = e^z = e^{x+iy} = e^x e^{iy}. \]

From this we see that the line segment \([ (a, c), (a, d) ] \) (the left side of the rectangle) gets mapped to the curve \( \{ r = e^a, c \leq \theta \leq d \} \). Similarly, the right side of the rectangle, the line segment \([ (b, c), (b, d) ] \), gets mapped to the curve \( \{ r = e^b, c \leq \theta \leq d \} \). The lower side gets mapped to the curve (in polar coordinates) \([ (e^a, c), (e^b, c) ] \), and the upper side \([ (a, d), (b, d) ] \) get mapped to the curve (in polar coordinates) \([ (e^a, d), (e^b, d) ] \).

Exercise 8 (page 43 of B&C). Indicate graphically the vector field represented by

a) \( \omega = iz \): Solution. The picture looks like this: take some point \( z \in \mathbb{C}, z \neq 0 \). Then the vector \( \omega(z) \), “attached” to this point is a vector that can be obtained from the vector \( z \) by rotating it by \( \pi/2 \) angle, counterclockwise.

b) \( \omega = z/|z| \): Solution. For every point \( z \), the vector \( \omega(z) \) attached at this point is the unit vector, directed out of the center of coordinates.