Assignment #9

DUE *Wednesday 28 November, 2007*

Exercise 1. Show that Fourier's choice,

$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\},$$

is an orthogonal set (sequence) in $L^2(-\pi, \pi)$. (*That is, show that the integral from $-\pi$ to π of the product of any pair of functions from this set is zero.*)

Exercise 2. The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are an orthogonal set in $L^2(-1, 1)$ (that is, on the interval $[-1, 1]$). Recall

$$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, dx$$

is the inner product in this case. Note that the nth Legendre polynomial $P_n(x)$ is a polynomial of degree n. In fact,

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\
&\vdots
\end{aligned}$$

They happen to be normalized so that $P_n(1) = 1$, but this is not terribly important.

(a) Plot P_0, P_1, P_2, P_3 on the same axes, on the interval $[-1, 1]$.

(b) Find P_4 this way: P_4 is a polynomial of degree four. Therefore $f(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ is a multiple of P_4 if these facts are true: $(P_0, f) = 0$, $(P_1, f) = 0$, $(P_2, f) = 0$, $(P_3, f) = 0$. Find c_3, c_2, c_1, c_0 using these facts. (That is, set up and solve a very easy linear system of four equations in four unknowns.) Now normalize, that is, compute $f(1)$ and divide: $P_4(x) = f(x)/f(1)$.

The final answer to this part is $P_4(x)$ written out explicitly.

(c) Add P_4 to your plot in part **(a)**.

Exercise 3. **(a)** The set

$$\left\{ \frac{1}{\sqrt{2\pi}}\, e^{inx} \right\}_{n=-\infty}^{\infty}$$

is ortho*normal* in $L^2(-\pi, \pi)$. (I showed this fact in class.) Use this fact to find the coefficients a_n in the Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, \frac{1}{\sqrt{2\pi}}\, e^{inx}$$

if $f(x) = |x|$ on the interval $[-\pi, \pi]$. Recall that $(f, g) = \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$.

(b) Using the identities

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right), \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

as necessary, rewrite the series in part **(a)** as a Fourier cosine series with real coefficients. (*This will eliminate all complex number considerations in the rest of the problem.*)

Could you know in advance that the coefficients of $\sin(nx)$ are zero?

(c) As pointed out in class, we are finding a series for the periodic extension of $f(x) = |x|$ to the whole real line, with period 2π. Let $\bar{f}(x)$ be this periodic function. Plot $\bar{f}(x)$ and the derivative of $\bar{f}(x)$ on the same axes.

(d) Differentiate the Fourier cosine series in part **(b)** to get a Fourier sine series for the derivative of $\bar{f}(x)$. Compare to the solution to exercise #3 in Lesson 5 in the textbook; explain why these are essentially the same Fourier sine series.

Exercise 4. **(a)** Let $g(x) = |\sin(x/2)|$ on the interval $[-\pi, \pi]$. Graph $g(x)$ on $[-\pi, \pi]$. On separate axes, graph the periodic extension of $g(x)$ with period 2π.

(b) Find the Fourier series of $g(x)$ using the "Fourier's choice" series in problem **1**. I claim that because $g(x)$ is even you will get

$$g(x) \overset{\circledast}{=} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for some coefficients a_n; the coefficients of $\sin(nx)$ will be zero. Confirm this, that is, confirm that the coefficients of $\sin(nx)$ are zero and then find all a_n.

(c) Considering the graph of the periodic extension of $g(x)$, where will convergence of the series be poorest? Compare the locations which are odd multiples of π with those that are even multiples of π.

(d) Use a computer to plot, on the same axes, $g(x)$ and the fifth partial sum of $\circledast$, that is, where the upper limit of the sum in $\circledast$ is 4. Do the same for the ninth partial sum of $\circledast$. Comment on the convergence of the series at integer multiples of π.

Hint on **4**: *If, instead of* $g(x) = |\sin(x/2)|$ *above, we have* $h(x) = |\sin(x)|$*, then*

$$h(x) = \frac{4}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{8}{1-4k^2}\, \cos(2kx).$$

is a Fourier series; again $h(x)$ *is even so it is a Fourier cosine series. Below is some* MATLAB *that does the plotting for the ninth partial sum, and its graphical output. The* MATLAB *program is also online at* `www.dms.uaf.edu/~bueler/Math421F07.htm`.

```
% PLOTABSSIN  Plot h(x)=|sin(x)|, and the ninth partial sum of its Fourier
%             series, on the interval [-4,4].

% build partial sum:
x = linspace(-4,4,2001);                          % Matlab needs x coordinates to plot
C = 1/(2*pi);                                     % normalization constant
y = (4) * C*ones(size(x));                        % constant term
for k=1:4                                         % n=2k; thus n=0,1,2,...,8
    y = y + (8/(1-4*k*k)) * C*cos(2*k*x);
end

plot(x,abs(sin(x)),'--',x,y)
legend('h(x)','ninth partial sum')
set(gca,'XTick',[-pi 0 pi])                       % shows interval [-pi,pi] clearly
set(gca,'XTickLabel',['-pi'; ' 0 '; 'pi '])       % mere beautification
```

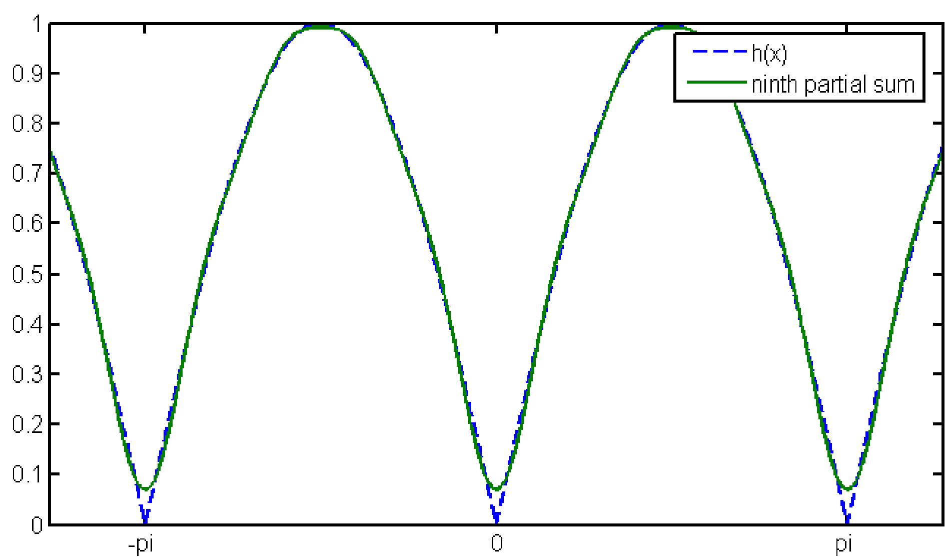

FIGURE 1. The function $h(x) = |\sin(x)|$ and a partial sum of its Fourier cosine series. For this function, convergence is poor at every integer multiple of π.