Selected Solutions to Assignment \#8

Total of 18 points.

Exercise I: (6 pts) I give only the high points

**Separation-of-variables.**

\[ u(x, t) = X(x)T(t) \text{ and PDE} \quad \implies \quad \frac{c^2T''}{T} = -\lambda^2 = \frac{X''}{X}. \]

**Eigenvalue problem and its solution.**

\[ X'' + \lambda^2 X = 0, \ X(0) = 0, \ X(L) = 0 \quad \implies \quad \lambda_j = \frac{j\pi}{L}, \ X_j(x) = \sin(j\pi x/L). \]

**T(t) problems and its solution.**

\[ c^2T'' + \lambda^2 T = 0 \quad \implies \quad T(t) = c_1 \cos(\lambda t/c) + c_2 \sin(\lambda t/c). \]

**Expansion.**

\[ u(x, t) = \sum_{j=1}^{\infty} \left[ a_j \cos(\lambda_j t/c) + b_j \sin(\lambda_j t/c) \right] \sin(\lambda_j x). \]

**Orthogonality.**

\[ \int_0^L \sin(\lambda_j x) \sin(\lambda_k x) \, dx = \begin{cases} 0, & j \neq k, \\ \frac{L^2}{\pi^2}, & j = k. \end{cases} \]

**Homogeneous initial condition.**

\[ 0 = u(x, 0) = \sum_{j=1}^{\infty} [a_j \cdot 1 + b_j \cdot 0] \sin(\lambda_j x) \quad \implies \quad a_j = 0. \]

**Non-homogeneous initial condition.**

\[ g(x) = u_t(x, 0) = \sum_{j=1}^{\infty} b_j \frac{j\pi}{cL} \sin(\lambda_j x) \quad \implies \quad b_j = \frac{2c}{j\pi} \int_0^L g(x) \sin \left( \frac{j\pi x}{L} \right) \, dx. \]

**Final answer.**

\[ u(x, t) = \sum_{j=1}^{\infty} b_j \sin \left( \frac{j\pi t}{cL} \right) \sin \left( \frac{j\pi x}{L} \right) \]

along with the boxed formula for \( b_j \).

Exercise II: (Not graded.)

**Separation-of-variables.**

\[ u(x, t) = X(x)T(t) \text{ and PDE} \quad \implies \quad \frac{T''}{T} = -\lambda^2 = \frac{X''}{X}. \]

**Eigenvalue problem and its solution.**

\[ X'' + \lambda^2 X = 0, \ X(0) = 0, \ X'(1) = 0 \quad \implies \quad \lambda_j = (2j-1)\pi, \ X_j(x) = \sin(\lambda_j x), \ j = 1, 2, 3, \ldots. \]

**T(t) problems and its solution.**

\[ T'' + \lambda^2 T = 0 \quad \implies \quad T(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t). \]
Expansion.

\[ u(x, t) = \sum_{j=1}^{\infty} [a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)] \sin(\lambda_j x). \]

Orthogonality.

\[ \int_0^1 \sin(\lambda_j x) \sin(\lambda_k x) \, dx = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}, & j = k. \end{cases} \]

Homogeneous initial condition.

\[ 0 = u_t(x, 0) = \sum_{j=1}^{\infty} [-a_j \lambda_j \cdot 0 + b_j \lambda_j \cdot 1] \sin(\lambda_j x) \implies b_j = 0. \]

Non-homogeneous initial condition.

\[ 5 \cos((\pi/2)x) = u(x, 0) = \sum_{j=1}^{\infty} a_j \sin(\lambda_j x) \implies a_j = 10 \int_0^1 \cos((\pi/2)x) \sin((2j-1)\pi x/2) \, dx. \]

Actual integral computation. Using \( \cos a \sin b = \frac{1}{2} (\sin(a + b) - \sin(a - b)) \),

\[ \int_0^1 \cos(\pi x/2) \sin((2j-1)\pi x/2) \, dx = \frac{1}{2} \int_0^1 \sin(2j\pi x/2) - \sin(2(1-j)\pi x/2) \, dx \\
= \frac{1}{2} \int_0^1 \sin(j\pi x) + \sin((j-1)\pi x) \, dx \\
= \frac{1}{2} \begin{cases} (j-1)^{-1}, & j \text{ odd,} \\ j^{-1}, & j \text{ even.} \end{cases} \]

Final answer.

\[ u(x, t) = \frac{10}{\pi} \sum_{j=1}^{\infty} \eta_j \cos((2j-1)\pi t/2) \sin((2j-1)\pi x/2) \]

where \( \eta_j = j^{-1} \) if \( j \) is even and \( \eta_j = (j-1)^{-1} \) if \( j \) is odd.

The question was intended to be simpler, with \( u(x, 0) = 5 \sin(\pi x/2) \). With the actual initial condition, the request “Sketch \( u(x, t) \) at several times” is impractical, as well as the request to give a physical interpretation.

Lesson 7, #1: (Not graded.) Answer in back of text.

Lesson 7, #2: (3 pts) First, \( p(x) = 1, q(x) = 0, \) and \( r(x) = 1 \) in the text’s form of the

\[ \lambda_j = ((2j-1)\pi/2)^2, \quad X_n(x) = \sin((2j-1)\pi x/2), \quad j = 1, 2, 3, \ldots \]

(Note: The text uses “\( \lambda \)” where I tend to write “\( \lambda^2 \)” in class. The solution here matches the problem statement in the text, of course.)

Lesson 7, #3: (6 pts) [Recommendation: read solution to #4 first.]

Separation-of-variables.

\[ u(x, t) = X(x)T(t) \text{ and PDE} \implies \frac{T''}{T} = -\lambda = \frac{X''}{X}. \]
Eigenvalue problem and its solution.

\[ X'' + \lambda X = 0, \ X'(0) = 0, \ X'(1) = 0 \quad \implies \quad \lambda_j = (j\pi)^2, \ X_j(x) = \cos(j\pi x), \ j = 0, 1, 2, 3, \ldots \]

\(T(t)\) problems and its solution.

\[ T' + \lambda T = 0 \quad \implies \quad T(t) = e^{-\lambda t}. \]

Expansion.

\[ u(x, t) = a_0 + \sum_{j=1}^{\infty} a_j e^{-(j\pi)^2 t} \cos(j\pi x). \]

Orthogonality.

\[ \int_0^1 \cos(j\pi x) \cos(k\pi x) \, dx = \begin{cases} 0, & j \neq k, \\ 1, & j = k = 0, \\ \frac{1}{2}, & j = k > 0 \end{cases}. \]

Non-homogeneous initial condition.

\[ x = u(x, 0) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\pi x) \quad \implies \quad a_j = \begin{cases} \int_0^1 x \, dx, & j = 0, \\ \frac{1}{2} \int_0^1 x \cos(j\pi x) \, dx, & j = 0, \\ 0, & j > 0 \text{ even}, \\ -\frac{4}{(j\pi)^2}, & j > 0 \text{ odd}. \end{cases} \]

Final answer.

\[ u(x, t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{j=1}^{\infty} e^{-(2j-1)^2 t} \sin((2j-1)\pi x) / (2j-1)^2. \]

[Note: The solution in the back is in error in that it does not have zero coefficients when the index is even.]

Lesson 7, #4: (3 pts) Here we should start with considering cases \( \lambda < 0, \ \lambda = 0, \) and \( \lambda > 0. \) The boundary conditions do not allow nontrivial solutions if \( \lambda > 0. \) For \( \lambda = 0, \)

\[ X(x) = c_1 x + c_2 \ & \text{BCs} \quad \implies \quad X(x) = 1, \]

while for \( \lambda < 0, \)

\[ X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \ & \text{BCs} \quad \implies \ \lambda_j = (j\pi)^2, \ X_j(x) = \cos(j\pi x), \ j = 1, 2, 3, \ldots \]

Thus the eigenvalues are \( \lambda_j = (j\pi)^2, \ j = 0, 1, 2, \ldots, \) and \( X_j(x) = \cos(j\pi x), \ j = 0, 1, 2, \ldots \)