Assignment #6

Due Monday 8 November at start of class.

Read section 3.5 on cubic splines in the textbook (Burden & Faires, 9th ed.).

Recall that the following theorem was proven in class, even though it is not clearly-stated in the text:

Theorem. Suppose that $f \in C^2[a,b]$ and that $n+1$ distinct interpolation points x_j are given in order: $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Let $\Delta x = \max_{j=1,\dots,n}(x_j - x_{j-1})$ and let $M = \max_{x\in[a,b]} |f''(x)|$. Let $R(x)$ be the piecewise-linear interpolant of $f(x)$ at the points $x_0, \dots, x_n$. (*Thus $R(x_j) = f(x_j)$ at $j = 0, \dots, n$, and $R(x)$ is a linear function on each interval* $[x_{j-1}, x_j]$.) Then, for every $x \in [a,b]$,

$$|R(x) - f(x)| \le \frac{M}{2}\Delta x^2.$$

1. For each of the following functions and interpolation points, plot $f(x)$ and the piecewise-linear interpolant $R(x)$ on the same axes. Then apply the above theorem to compute an upper bound on the error $|R(x) - f(x)|$.

(a) $f(x) = \cos(x), \qquad x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

(b) $f(x) = \exp(-x^2), \qquad x_0 = 0, x_1 = 0.5, x_2 = 1.2, x_4 = 3.0, x_5 = 5.0$

2. The *(composite) trapezoid rule* can be described by the following simple summary: Given $f(x)$ defined on $[a,b]$, and a positive integer n. Let $\{x_j\}$ be the $n+1$ equally spaced points with spacing $\Delta x = (b-a)/n$. Compute the equally-spaced piecewise-linear interpolant $R(x)$ of $f(x)$ for these points. Integrate $R(x)$ to approximate the integral of $f(x)$:

$$\int_a^b f(x)\,dx \approx \int_a^b R(x)\,dx.$$

By using the triangle inequality, namely

$$\left|\int_a^b R(x)\,dx - \int_a^b f(x)\,dx\right| \le \int_a^b |R(x) - f(x)|\,dx,$$

and by using the theorem at the top of this page, show the following error estimate for the trapezoid rule:

$$\left|\int_a^b R(x)\,dx - \int_a^b f(x)\,dx\right| \le \frac{M(b-a)^3}{2n^2}$$

where $M = \max_{x\in[a,b]} |f''(x)|$.

3. *Do both parts of this exercise "by hand". That is, show all the steps of constructing the cubic spline by finding its coefficients.* But *you may use* MATLAB/OCTAVE *or other calculator to do individual arithmetic operations. It is recommended that you use this form for the the cubic polynomials that form the parts of the spline:* $S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$.

Compute and plot the natural cubic spline for the following data:

(a)

x	$f(x)$
0	1.00000
0.5	2.71828

(b)

x	$f(x)$
0.1	-0.29004996
0.2	-0.56079734
0.3	-0.81401972

4. *Do both parts of this exercise using the program*

`http://www.dms.uaf.edu/~bueler/ncspline.m`

Note that you can compare your answer in part **(a)** *here to the result from part* **(b)** *of the previous exercise.*

Compute and plot the natural cubic spline for the following data:

(a)

x	$f(x)$
0.1	-0.29004996
0.2	-0.56079734
0.3	-0.81401972

(b)

x	$f(x)$
0.1	0.98020
0.2	0.92312
0.3	0.83527
0.4	0.72615
0.5	0.60653
0.6	0.48675
0.7	0.37531

5. The data in part **(b)** of exercise **3**, and part **(a)** of exercise **4** also, is just a table of the values of

$$f(x) = x^2 \cos x - 3x.$$

Use the result of those parts of previous problems, namely the cubic spline computed from the data, to approximate $f(0.18)$ and $f'(0.18)$. Give the actual errors.