Solutions to Assignment #4

1. a. $f(p) = p^4 + 2p^2 - p - 3 = 0$ can be re-arranged to

$$p^4 = 3 + p - 2p^2 \qquad \text{or} \qquad p = \left(3 + p - 2p^2\right)^{1/4}$$

Thus $f(p) = 0$ if and only if p is a fixed point of $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$. Here are 20 iterations:

```
>> g1 = @(x) (3+x-2*x^2)^(1/4)
>> format long g
>> p = 1, for n=1:20, p = g1(p), end
p =                    1
p =     1.18920711500272
p =     1.08005775266756
...
p =     1.12410508074685
p =     1.12413407454347
p =     1.12411623301991
```

This at least looks like it is converging to a fixed point.

b. Similarly, $f(p) = 0$ can be rewritten

$$2p^2 = p + 3 - p^4 \qquad \text{or} \qquad p = \left(\frac{p + 3 - p^4}{2}\right)^{1/2}$$

Thus $f(p) = 0$ if and only if p is a fixed point of $g_2(x) = \left((x + 3 - x^4)/2\right)^{1/2}$. Here are 20 iterations:

```
>> g2 = @(x) ((x+3-x^4)/2)^(1/2);
>> p = 1, for n=1:20, p = g2(p), end
p =                    1
p =     1.22474487139159
...
p =    0.957226754592156
p =     1.24852955693609
p =    0.953569842385412
p =     1.25035027839163
p =    0.950317401707698
```

This looks like some kind of oscillation, which is growing in magnitude.

c. Just from the above info, $g_1(x)$ seems more promising. (*Given the theory we know, we should expect that* $|g_1'(1.1241)| < 1$ *while* $|g_2'(1.1241)| > 1$. *You can check that this is true.*)

2. a. First, $g'(x) = \cos x$ so on the interval $[2, 3]$, we know $g(x)$ is decreasing because $\cos x < 0$ on this interval. Because $g(x)$ is decreasing on this interval we need only check that $g(2)$ and $g(3)$ are in the interval $[2, 3]$ in order to know that $g(x)$ is in $[2, 3]$ for all $x \in [2, 3]$. But $g(2) = 2.9093$ while $g(3) = 2.1411$. Thus $g(x) \in [2, 3]$ if $x \in [2, 3]$.

Now,

$$\max_{2 \le x \le 3} |g'(x)| = \max_{2 \le x \le 3} |\cos x| = |\cos 3| = 0.98999.$$

Let $k = 0.99 < 1$. Because g is continuous and $|g'(x)| \le k < 1$ on the interval $[2,3]$, by theorem 2.3 there is a unique fixed point on this interval. Furthermore we see that by theorem 2.4 the iteration $p_n = g(p_{n-1})$ will converge for any starting point p_0 in the interval $[2,3]$.

Finally, it is easy to see that $f(p) = 2 + \sin p - p = 0$ can be manipulated to $p = g(p) = 2 + \sin p$.

b. Here $g'(x) = (2/3)\,(2x+5)^{-1/3}$. This is always positive if $x \in [2,3]$ so $g(x)$ is increasing. But $g(2) = 2.0801$ and $g(3) = 2.2240$ so $g(x) \in [2,3]$ if $x \in [2,3]$. On the other hand,

$$\max_{x\in[2,3]} |g'(x)| = \max_{x\in[2,3]} \frac{2}{3}\,(2x+5)^{-1/3} = \frac{2}{3}\left(\min_{x\in[2,3]} 2x+5\right)^{-1/3} = \frac{2}{3}\,(2(2)+5)^{-1/3} = 0.32050.$$

Let $k = 0.33$. Then $|g'(x)| \le k < 1$ if $x \in [2,3]$ so by theorem 2.3 there is a unique fixed point $p = g(p)$ and by theorem 2.4 the iteration $p_n = g(p_{n-1})$ will converge for any starting point $p_0 \in [2,3]$. Finally, it is easy to see that $f(p) = p^3 - 2p - 5$ can be manipulated to $p^3 = 2p+5$ or $p = g(p) = (2p+5)^{1/3}$.

3. A function with the desired properties could be discontinuous, and have slope greater than one (in magnitude) at the fixed point, but still have only one fixed point. For example,

$$g(x) = \begin{cases} 1 - (x/0.6), & 0 \le x \le 0.6, \\ 0.2, & 0.6 < x \le 1. \end{cases}$$

Here $g(x)$ is defined for all $x \in [0,1]$, and $g(x) \in [0,1]$ if $x \in [0,1]$, but g has the properties just mentioned. See Figure 1. The figure is generated by the next code, which can only be of interest as an illustration of plotting commands.

uniquefixed.m

```
% UNIQUEFIXED  Plot piecewise linear function.

clf,  x=0:0.001:0.6;  plot(x,1.0 - (1.0/0.6)*x)
hold on,  x=0.6:.001:1.0;  plot(x,0.2*ones(size(x)))
plot([0 0.6 1.0],[1.0 0.0 0.2],'o','markersize',6,'linewidth',6.0)
plot(0.6,0.2,'o','markersize',10,'linewidth',1.5)
x=0:0.001:1.0;  plot(x,x,'g')
axis([0 1 0 1]),  xlabel x,  ylabel y,  hold off,  axis equal
% print -dpdf uniquefixed.pdf
```

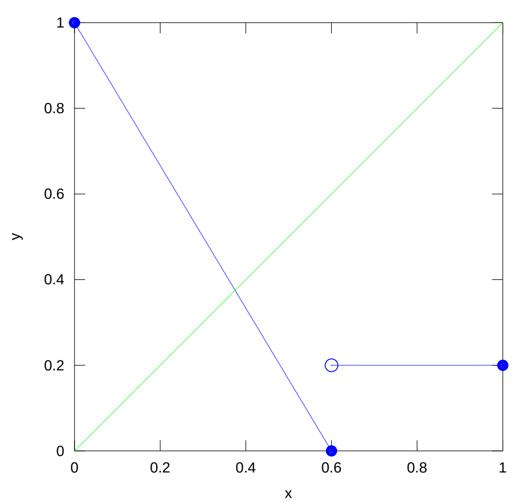

FIGURE 1. Plot of $y = g(x)$ for a piecewise constant $g(x)$ which is not continuous and which has slope greater than one in magnitude at the fixed point $p = g(p)$.

4. Here $g(x) = (x/2) + 7/(2x)$. The number p is a fixed point of g if and only if the following equivalent statements are true:

$$p = \frac{1}{2}p + \frac{7}{2p} \qquad \text{or} \qquad 2p - p = \frac{7}{p} \qquad \text{or} \qquad p^2 = 7$$

or $p = \pm\sqrt{7}$. Consider the derivative at the positive fixed point:

$$g'(x) = \frac{1}{2} - \frac{7}{2x^2} \qquad \text{so} \qquad g'(\sqrt{7}) = \frac{1}{2} - \frac{7}{2(7)} = 0.$$

In fact, for any $2 \le x \le \sqrt{7}$ we have

$$|g'(x)| = \left|\frac{1}{2} - \frac{7}{2x^2}\right| = \frac{1}{2}\left|1 - \frac{7}{x^2}\right| = \frac{1}{2}\left(\frac{7}{x^2} - 1\right) \le \frac{1}{2}\left(\frac{7}{2^2} - 1\right) = \frac{3}{8}$$

while for any $x \ge \sqrt{7}$ we have

$$|g'(x)| = \frac{1}{2}\left(1 - \frac{7}{x^2}\right) \le \frac{1}{2}(1 - 0) = \frac{1}{2}.$$

Thus on the interval $[2, \infty]$ we have $|g'(x)| = \frac{1}{2}$. Also $g(x) \in [2, \infty)$ if $x \in [2, \infty)$. Thus by theorem 2.4 there is a unique fixed point on the interval $[2, \infty$, which we already know is $p = \sqrt{7}$, and the iteration $x_n = g(x_{n-1})$ converges to it for any $x_0 \ge 2$.

Comment. Consider the root-finding problem $f(x) = x^2 - 7 = 0$. Apply Newton's method:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{x_{n-1}^2 - 7}{2x_{n-1}} = \frac{2x_{n-1}^2}{2x_{n-1}} - \frac{x_{n-1}^2 - 7}{2x_{n-1}} = \frac{1}{2}x_{n-1} + \frac{7}{2x_{n-1}}.$$

Thus the above argument shows Newton's method works (converges) for any starting point $x_0 \ge 2$.

5. Here Newton's method is:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{-p_{n-1}^3 - \cos(p_{n-1})}{-3p_{n-1}^2 + \sin(p_{n-1})}.$$

Using MATLAB/OCTAVE to find p_2 given $p_0 = -1$:

```
>> g = @(x) x - (-x^3-cos(x))/(-3*x^2+sin(x));
>> p = -1;
>> p = g(p)
p = -0.88033
>> p = g(p)
p = -0.86568
```

Thus $p_2 = -0.86568$, approximately. If we attempt $p_0 = 0$ we get:

```
>> p = 0; p = g(p)
warning: division by zero
p = Inf
```

Of course, the reason for difficulties should be obvious: $f'(0) = 0$.

6. In these exercises I start by checking I have a bracket on the given interval. We see the secant method converges almost as fast as Newton's. I know I have 10^{-5} accuracy because the iterations agree to 14 digits.

a.

```
>> f = @(x) exp(x) - 2.^(-x) + 2 * cos(x) - 6;
>> [f(1)  f(2)]
ans =
     -2.70111355980468     0.306762425836365
>> df = @(x) exp(x) + log(2) * 2.^(-x) - 2 * sin(x);
>> p = 1.8,  for n=1:5, p = p - f(p) / df(p), end               % Newton's
p =                  1.8
p =     1.96087660418324
p =     1.94465320885678
p =     1.94446250759735
p =     1.94446248157493
p =     1.94446248157493
>> polder=1.8;  pold=1.9                                        % Secant
pold =                  1.9
>> for n=1:6, pnew = pold - (pold-polder) * f(pold) / (f(pold)-f(polder)); ...
>   polder=pold;  pold=pnew,  end
pold =     1.94934738830449
pold =     1.94430510658999
pold =     1.94446193240902
pold =     1.94446248163677
pold =     1.94446248157493
pold =     1.94446248157493
```

b.

```
>> f = @(x) log(x-1) + cos(x-1);
>> [f(1.3)  f(2)]
ans =
     -0.24863631520033      0.54030230586814
>> df = @(x) 1./(x-1) - sin(x-1);
>> p = 1.6,  for n=1:6,  p = p - f(p)/df(p),  end         % Newton's
p =                  1.6
p =     1.31460699949791
p =     1.38623612050177
p =     1.39752389069251
p =      1.3977483900825
p =     1.39774847595873
p =     1.39774847595875
>> pold = 1.6;  p = 1.4                                   % Secant
p =                  1.4
>> for n=1:6,  pnew = p - f(p) * (p-pold) / (f(p)-f(pold));  pold=p;  p=pnew,  end
p =     1.39691982546514
p =     1.39775165385563
p =     1.39774848044192
p =     1.39774847595872
p =     1.39774847595875
p =     1.39774847595875
```

7. These exercises were adequately discussed in class.

8. Likewise.