Assignment #4

Due Monday 11 October at start of class.

Read sections 2.2, 2.3, and 2.4 of the textbook Burden & Faires. The exercises below are similar to certain ones in those sections.

1. Let $f(x) = x^4 + 2x^2 - x - 3$. This exercise is about fixed point iterations which might solve $f(x) = 0$.

(a) Use algebraic manipulation to show that the function

$$g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$$

has a fixed point p precisely when $f(p) = 0$. Now perform 20 fixed point iterations $p_n = g_1(p_{n-1})$, starting with $p_0 = 1$. (*The latter is easiest to do with* MATLAB/OCTAVE, *of course. In any case, show some work.*)

(b) Do the same things with

$$g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}.$$

(c) Based on the above evidence, and without doing any further analysis, which of the two fixed point iterations is more promising to solve $f(x) = 0$?

2. For each of the following functions g and intervals, show using theorem 2.3 that there is a unique fixed point. Show using theorem 2.4 that a fixed point iteration starting with any p_0 in the given interval will converge. Finally, show that a fixed point of g solves $f(p) = 0$ for the given function f.

(a) $g(x) = 2 + \sin x, \quad [2, 3]; \qquad f(x) = 2 + \sin x - x$

(b) $g(x) = (2x + 5)^{1/3}, \quad [2, 3]; \qquad f(x) = x^3 - 2x - 5$

3. Find a specific function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.3 but which still has a unique fixed point on $[0, 1]$. (*You might start by graphing an example $y = g(x)$. Then find a simple formula which works for that picture.*)

4. Show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{7}{2x_{n-1}},$$

for $n \geq 1$ and with any $x_0 \geq 2$, converges to $\sqrt{7}$. (*Hints: Identify g so that the sequence is exactly $x_n = g(x_{n-1})$. Show that a fixed point of g is a square root of 7. Then apply theorem 2.4 to this fixed point iteration. Specifically, you will find the maximum and minimum of $g'(x)$ on the interval $[2, \infty)$ because you seek a bound on $|g'(x)|$. You will see that the location of the right endpoint of the interval "$[a, b]$" in the theorem does not really matter in this case.*)

Historical Note: The rule

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}$$

is called the *Babylonian rule* or *mechanic's rule* for approximating $\sqrt{A}$.

It can described in words as: "*To find the square root of A, take a guess. Then average together the guess and A over the guess. This is a new guess. Repeat.*" For example, to approximate $\sqrt{5}$ we guess 2 and compute a new guess as the average of 2 and $5/2 = 2.5$, namely 2.25. And etc.

It is a *very good* scheme, especially if x_0 is a remotely-decent first guess. Why? It turns out that it is merely Newton's method on the equation $x^2 - A = 0$.

5. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?

6. For each of the following equations, use both Newton's method and the Secant method to find solutions accurate to within 10^{-5}. (*Start by explaining why a solution exists on the given interval. Also, make sure to clearly state the starting value, or values, which you choose for your iterations. State why you think you have* 10^{-5} *accuracy. Show any* MATLAB/OCTAVE *codes used.*)

(a) $e^x - 2^{-x} + 2\cos x - 6 = 0, \qquad [1, 2]$

(b) $\ln(x-1) + \cos(x-1) = 0, \qquad [1.3, 2]$

7. Show that the following sequences converge linearly to $p = 0$. How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$?

(a) $p_n = \frac{1}{n}, \quad n \geq 1$

(b) $p_n = \frac{1}{n^2}, \quad n \geq 1$

8. **(a)** Show that for any positive integer k, the sequence defined by $p_n = 1/n^k$ converges linearly to $p = 0$.

(b) Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to $p = 0$.