# **Selected Solutions to Assignment #1** (with corrections)

**2.** *Because the functions are continuous on the given intervals, we show these equations have solutions by checking that the functions have opposite signs at the ends of the given intervals, and then by invoking the Intermediate Value Theorem (IVT).*

**a.** $f(0.2) = -0.28399$ and $f(0.3) = 0.0066009$. Because $f(x)$ is continuous on $[0.2, 0.3]$, and because $f$ has different signs at the ends of this interval, by the IVT there is a solution $c$ in $[0.2, 0.3]$ so that $f(c) = 0$.

Similarly, for $[1.2, 1.3]$ we see $f(1.2) = 0.15483, f(1.3) = -0.13225$. And etc.

**b.** Again the function is continuous. For $[1, 2]$ we note $f(1) = 1$ and $f(2) = -0.69315$. By the IVT there is a $c$ in $(1, 2)$ so that $f(c) = 0$.

For the interval $[e, 4]$, $f(e) = -0.48407$ and $f(4) = 2.6137$. And etc.

**3.** *One of my purposes in assigning these was that you should recall the* Extreme Value Theorem. *It guarantees that these problems have a solution, and it gives an algorithm for finding it. That is, "search among the critical points and the endpoints."*

**a.** The discontinuities of this rational function are where the denominator is zero. But the solutions of $x^2 - 2x = 0$ are at $x = 2$ and $x = 0$, so the function is continuous on the interval $[0.5, 1]$ of interest. So we find the derivative by the quotient rule; after simplification:

$$f'(x) = -\frac{2\left(2x^2 + 5x + 3\right)}{(x^2 - 2x)^2}.$$

This derivative is zero where the numerator is zero:

$$2x^2 + 5x + 3 = 0.$$

Because I did not see the roots right away, I used the quadratic formula:

$$x = \frac{-5 \pm \sqrt{25 - 24}}{4} = \{-1, -3/2\}.$$

But neither of the roots is in the interval $[0.5, 1]$. Thus we evaluate the endpoints, and the solution must be one of these:

$$f(0.5) = 4/3, \qquad f(1) = -1.$$

Thus the maximum of $f(x)$ on $[0.5, 1]$ occurs at $x = 0.5$ and has value $4/3$.

**b.** We might as well plug in the endpoints first: $f(2) = -2.6146$, $f(4) = -5.1640$. But a graph shows a maximum in the interior, actually. One way to pin down its location is, as usual, to find $f'(x)$ and solve $f'(x) = 0$. After some simplification,

$$f'(x) = 2\left(\cos(2x) - x\sin(2x) - x + 2\right).$$

I do not know how to solve $f'(x) = 0$ by hand. But it is pretty easy to graph the derivative and "zoom in" to get close to the solution; one way looks like this:

```
>> dfdx = @(x) cos(2*x) - x .* sin(2*x) - x + 2;
>> x=2:0.001:4;  plot(x,dfdx(x)),  grid on
>> x=3.1:0.0001:3.15;  plot(x,dfdx(x)), grid on
>> x=3.12:0.0001:3.125;  plot(x,dfdx(x)), grid on
>> x=3.1218:0.0001:3.1222;  plot(x,dfdx(x)), grid on
>> x=3.1219:0.00001:3.12195;  plot(x,dfdx(x)), grid on
>> dfdx(3.121935)
ans =  4.6913e-07
```

Note that $f(3.121935) = 4.9803$, and this is (close to) the maximum of $f(x)$.

I would guess that I just got a 5 digit-accurate estimate of the location of the maximum of $f$. *There are more systematic ways* to solve equations like $f'(x) = 0$, including the robust bisection method, which would work just fine here.

**4.** *to show:* Suppose $f \in C[a,b]$ and $f'(x)$ exists on $(a,b)$. If $f'(x) \neq 0$ for all $x$ in $(a,b)$, then there exists at most one number $p$ in $[a,b]$ with $f(p) = 0$.

PROOF. The assumptions about $f$ allow us to apply the Mean Value Theorem (MVT). Let us suppose, to see what consequences follow, that there are (at least) two solutions: $f(p_1) = 0, f(p_2) = 0, p_1 \neq p_2$. By the MVT there is $c$ in $[p_1, p_2]$ so that

$$f'(c) = \frac{f(p_2) - f(p_1)}{p_2 - p_1} = \frac{0}{p_2 - p_1} = 0.$$

But the existence of a zero of $f'$ contradicts the hypotheses we are assuming; they say there is no such root. Thus there cannot be two solutions, but at most one. □

*Comment 1. It is perfectly-possible that there are no solutions, even for functions $f$ satisfying the hypotheses. Let $f(x) = x$ and $[a,b] = [1,2]$, for example.*

*Comment 2. You may also prove this by the "special MVT", namely Rolle's theorem.*

**5.** Let $f(x) = \sqrt{x+1}$ and $x_0 = 0$. Then $f(x_0) = 1$ and

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} \quad \text{so} \quad f'(x_0) = \frac{1}{2},$$
$$f''(x) = -\frac{1}{4}(x+1)^{-3/2} \quad \text{so} \quad f''(x_0) = -\frac{1}{4},$$
$$f'''(x) = \frac{3}{8}(x+1)^{-5/2} \quad \text{so} \quad f'''(x_0) = \frac{3}{8}.$$

Thus

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3. \end{aligned}$$

Figure 1 plots both functions, and we see $P_3$ approximates $f$ quite well near zero, as expected. By the way, Figure 1 was produced by these MATLAB/OCTAVE commands, which use "anonymous" functions, the code with the "@" symbol:

```
>> f = @(x) sqrt(x+1)
>> P3 = @(x) 1 + 0.5*x - (1/8)*x.^2 + (1/16)*x.^3
>> x=-1:0.001:1.5;  plot(x,f(x))
>> xlong=-1.5:0.001:1.5;  hold on,  plot(xlong,P3(xlong),'g'),  hold off
>> grid on,  xlabel x
```

Now we actually answer the question:

- $0.70711 = \sqrt{0.5} = f(-0.5) \approx P_3(-0.5) = 0.71094$, with actual error
$$|P_3(-0.5) - f(-0.5)| = 3.8 \times 10^{-3}$$
- $0.86603 = \sqrt{0.75} = f(-0.25) \approx P_3(-0.25) = 0.86621$, with actual error
$$|P_3(-0.25) - f(-0.25)| = 1.9 \times 10^{-4}$$
- $1.11803 = \sqrt{1.25} = f(0.25) \approx P_3(0.25) = 1.11816$, with actual error
$$|P_3(0.25) - f(0.25)| = 1.3 \times 10^{-4}$$
- $1.22474 = \sqrt{1.5} = f(0.5) \approx P_3(0.5) = 1.22656$, with actual error
$$|P_3(0.5) - f(0.5)| = 1.8 \times 10^{-3}$$

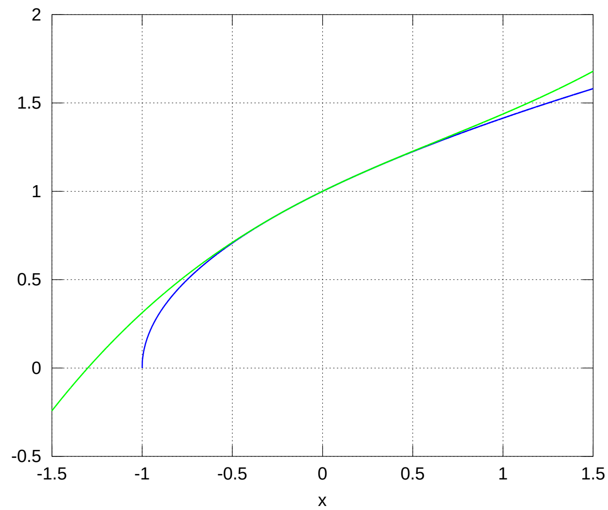


FIGURE 1. Plot of $f(x) = \sqrt{x+1}$ (blue) and $P_3(x)$ (green). Note that $P_3(x)$ is defined on the whole real line while $f(x)$ is only defined on $[-1, \infty)$. The two functions agree to screen resolution on about $(-0.3, 0.3)$.

*Comment. Stating the actual error with just two digits of precision is fine. Unlike the answer itself we generally don't need the error to be accurately known, as its magnitude matters most.*

**6.** We are given $f(x)$ and the basepoint. I compute and simplify the (eventually) needed derivatives by-hand:

$$\begin{aligned} f'(x) &= 2\cos(2x) - 4x\sin(2x) - 2(x-2), \\ f''(x) &= -8\sin(2x) - 8x\cos(2x) - 2, \\ f'''(x) &= -24\cos(2x) + 16x\sin(2x), \\ f^{(4)}(x) &= 64\sin(2x) + 32x\cos(2x), \\ f^{(5)}(x) &= -160\cos(2x) - 64x\sin(2x). \end{aligned}$$

**a.** For $P_3(x)$ I evaluate the derivatives through $f'''$ at $x_0 = 0$. Thus:

$$P_3(x) = -4 + 6x - x^2 - 4x^3.$$

I evaluate $P_3(0.4)$, and $f(0.4)$, and the actual error, by the following MATLAB/OCTAVE code, which again uses "anonymous functions":

```
>> f = @(x) 2*x.*cos(2*x) - (x-2).^2;
>> P3 = @(x) -4 + 6*x - x.^2 -4 * x.^3;
>> f(0.4)
ans = -2.0026
>> P3(0.4)
ans = -2.0160
>> abs(P3(0.4)-f(0.4))
ans =  0.013365
```

We see that the error in the approximation is better than 1 part in 100. In fact, the relative actual error is easily computable, too:

$$\frac{|P_3(0.4) - f(0.4)|}{|f(0.4)|} = 0.0067.$$

**b.** The remainder term uses the 4th derivative:

$$R_3(x) = \frac{64\sin(2\xi(x)) + 32\xi(x)\cos(2\xi(x))}{4!}(x-0)^4.$$

We are interested in the size of $R_3(0.4)$ because $|P_3(0.4) - f(0.4)| = |R_4(0.4)|$ is the error in using $P_3(0.4)$ to approximate $f(0.4)$. First,

$$R_3(0.4) = \frac{64\sin(2\xi(0.4)) + 32\xi(0.4)\cos(2\xi(0.4))}{24}(0.4)^4.$$

Now, all we know about the number "$\xi(0.4)$" is that it is between $x_0 = 0$ and $x = 0.4$. This is enough to estimate the numerator, in the last expression, using $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$:

$$|64\sin(2\xi(0.4)) + 32\xi(0.4)\cos(2\xi(0.4))| \le 64(1) + 32(0.4)(1) = 76.8.$$

It follows that

$$|P_3(0.4) - f(0.4)| = |R_4(0.4)| \le \frac{76.8}{24}(0.4)^4 = 0.082.$$

*This estimate of the absolute actual error does indeed exceed the absolute actual error we computed in part **a**.*

**c.** *Now I will be more brief.* Note $f^{(4)}(x_0) = 0$. That is, the 4th Taylor polynomial is the same as the third:

$$P_4(x) = -4 + 6x - x^2 - 4x^3.$$

Thus the actual error is *identical* to that in part **a**.

**d.** But the error *estimate* is better. We compute:

$$R_4(x) = \frac{160\cos(2\xi(x)) - 64\xi(x)\sin(2\xi(x))}{5!}(x-0)^5.$$

Thus

$$R_4(0.4) = \frac{160\cos(2\xi(0.4)) - 64\xi(0.4)\sin(2\xi(0.4))}{5!}(0.4)^5$$

and again we know $0 \le \xi(0.4) \le 0.4$ so

$$|160\cos(2\xi(0.4)) - 64\xi(0.4)\sin(2\xi(0.4))| \le 160 + 64(0.4) = 185.6$$

so

$$|P_4(0.4) - f(0.4)| = |R_4(0.4)| \le \frac{185.6}{120}(0.4)^5 = 0.016.$$

*Ah ha! This estimate is a very good upper bound on the absolute actual error we had in parts **a** and **c**, which was 0.013. The estimate is morally-superior, to a numerical analyst, because we did not need to know the exact value $f(0.4)$ to know the estimate, though one must know the function value to know the actual error.*

**7.** Let $x_0 = 0$; note we are interested in the small angle $1^\circ$, so this basepoint is natural. Note $f(0) = 0$. The derivatives of $f(x) = \sin x$ are

$$f'(x) = \cos x \qquad f'(0) = 1,$$
$$f''(x) = -\sin x \qquad f''(0) = 0,$$
$$f'''(x) = -\cos x \qquad f'''(0) = -1,$$

and so on. From these we see that

$$P_2(x) = 0 + 1\cdot x + 0 = x.$$

Thus the approximation $\sin x \approx x$ is actually a *quadratic* approximation. It is a quadratic function with zero "$x^2$" term. But then

$$R_2(x) = \frac{-\cos\xi(x)}{3!}x^3$$

and

$$|R_2(1^\circ)| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \frac{|\cos\xi(\pi/180)|}{6}\left(\frac{\pi}{180}\right)^3 \le \frac{1}{6}\left(\frac{\pi}{180}\right)^3 = 8.9\times 10^{-7}.$$

*So is* $\sin(\pi/180) \approx \pi/180$ *very accurate? The estimated error* $8.9\times10^{-7}$ *is small, but the number* $\pi/180$ *isn't that big either. This is a time to compute, and* approximate, *the relative error. Let's pretend we still don't know* $\sin(\pi/180)$. *In that case, here is what we know about the relative error:*

$$\frac{|\pi/180 - \sin(\pi/180)|}{|\sin(\pi/180)|} \approx \frac{|\pi/180 - \sin(\pi/180)|}{\pi/180} \le \frac{8.9\times10^{-7}}{\pi/180} = 5.1\times10^{-5}.$$

*So we expect that the approximation gets about the first 5 digits correct, and you will see by direct evaluation of* $\sin()$ *that this is true.*

**8.** **a.** $|(22/7)-\pi| = 1.2\times10^{-3}$, $\quad |(22/7)-\pi|/|\pi| = 4.0\times10^{-4}$

**b.** $|2.718-e| = 2.8\times10^{-4}$, $\quad |2.718-e|/|e| = 1.0\times10^{-4}$

**c.** $|40000-8!| = 320$, $\quad |40000-8!|/|8!| = 7.9\times10^{-3}$

**d.** $|\sqrt{2\pi}8^{8.5}e^{-8}-8!| = 417.6$, $\quad |\sqrt{2\pi}8^{8.5}e^{-8}-8!|/|8!| = 1.0\times10^{-2}$

**9.** **a.** Here $p^*$ is the truncated series and $p=e$. And

$$|p^*-p| = 1.6\times10^{-3}, \qquad \frac{|p^*-p|}{|p|} = 5.9\times10^{-4}.$$

**b.** Similarly,

$$|p^*-p| = 2.7\times10^{-8}, \qquad \frac{|p^*-p|}{|p|} = 1.0\times10^{-8}.$$

*I did all the computations in these lines of* Matlab/Octave:

```
>> p=exp(1);
>> n=0:5;  pstar=sum(1./factorial(n));  abs(pstar-p),  abs(pstar-p)/p
>> n=0:10;  pstar=sum(1./factorial(n));  abs(pstar-p),  abs(pstar-p)/p
```

*Note the use of* "`./`" *and of* "`sum`". *If it is not already clear, try it yourself, but look at the intermediate quantities!*

**10.** **I have corrected parts a and b.** *For 64 bit IEEE 754 representation we have this quick description of a machine number* $x$*:*

$$\text{sizes}: \quad s = 1 \text{ bit}, \quad c = 11 \text{ bits}, \quad f = 52 \text{ bits}.$$

$$\text{meaning}: \quad x = (-1)^s\, 2^{c-1023_{10}}\,(1+f), \qquad \text{with } f \text{ interpreted as a fraction, i.e. } 0.f_2$$

**a.** Here $c = 2^{10}+2^3+2 = 1034$ and

$$f = \frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256} = 0.57421875$$

so

$$x = (-1)^0 2^{1034-1023}\left(1+\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}\right) = +2^{11}\,(1.57421875) = 3224_{10}$$

**b.** There is merely a change in sign, thus: $x = -3224_{10}$.

**c.** Here $c = 2^{10}-1 = 1023$ and

$$f = \frac{1}{4}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256} = 0.32421875 \qquad \text{so} \qquad x = +2^0\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}\right) = 1.32421875_{10}$$

**d.** Here there is only a change in the last bit:

$$f = \frac{1}{4}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}+\frac{1}{2^{52}} = 0.3242187500000002220446$$

so $x = 1.3242187500000002220446_{10}$.