Selected Solutions to Assignment #2

(VERSION 2, WITH CORRECTIONS AND INCLUDING 3.1 #2)

Problems 1.1, exercise 2: *The problem asks you to do two limits.* To show that f is continuous at $x = 0$, compute $f(0) = 0$ (by definition) and note this is the same as

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0$$

(Justify by either the squeeze theorem or a picture, but give some indication of *why*.)

To show $f'(0)$ does not exist, compute the derivative from the definition:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0} \sin(1/h) \quad \text{d.n.e.}$$

(*Why* does the limit not exist? Again a picture suffices, in my opinion, or a proof can be done, looking at h for which $1/h$ is either a multiply of π or an odd multiple of $\pi/2$.)

Problems 1.1, exercise 6: By taking one sided limits we see $f(0) = \lim_{x\to 0} f(x)$, so f is continuous at $x = 0$. Because f is a polynomial near every other value of x, it is continuous everywhere. Again by taking one-sided limits we can show $f'(0)$ does not exist, and that $f'(x)$ has a jump at $x = 0$. Finally, because f' does not exist somewhere, we conclude f'' cannot exist there either.

Problems 1.1, exercise 11: Exactly this kind of problem is done in every calculus book to demonstrate the definition of "limit". Look there.

Problems 1.1, exercise 13: We are asked to find ξ so that

$$f(3) - f(1) = f'(\xi)(3-1)$$

where $f(x) = 3 - 2x + x^2$. This is equivalent to solving the equation

$$6 - 2 = (-2 + 2\xi)2$$

or $2 = -2 + 2\xi$ or $\xi = 2$. (Note $f'(x) = -2 + 2x$.)

Problems 1.1, exercise 15: The solution requires differentiating $f(x) = \cosh x$, finding the repeating pattern, plugging in $c = 0$ and finding the pattern in $f^{(k)}(0)$ and getting

$$\cosh x = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!}.$$

(This result can be checked by looking it up in any calculus book.)

Problems 1.1, exercise 23: Here we use $n = 2$ and $c = 0$ in Taylor's theorem with remainder to get the *equality*

$$\sin x = 0 + x + 0 + \frac{-\cos\xi}{6}x^3.$$

The error in the approximation $\sin x \approx x$ is the magnitude of the error term "$(-\cos(\xi)/6)x^3$". We can give an upper bound on this error term:

$$|\sin x - x| = \left|\frac{-\cos\xi}{6}x^3\right| \le \frac{1}{6}|x|^3$$

So we want to find x for which

$$\frac{1}{6}|x|^3 < 5 \times 10^{-7}.$$

(I am not too picky about meaning of "six decimal places," but I have used a good choice.) I get the interval $|x| \le 0.0144$ or $-0.0144 \le x \le 0.0144$. Thus we can trust "$\sin x \approx x$" to six digits as long as x is at most about $1/100$ radians.

Problems 1.1, exercise 24: Very similar to the above.

Problems 1.1, exercise 26: *Not graded.*

Problems 1.2, exercise 2: Two things to see before actually calculating. First that, by definition, we are to show

$$\lim_{n\to\infty} \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} = 0.$$

Second, because the values x_n are outputs of F applied to the previous values, we see $x_{n+2} - x_{n+1} = F(x_{n+1}) - F(x_n)$. But the MVT (p. 9) says $F(x_{n+1}) - F(x_n) = F'(\xi)(x_{n+1} - x_n)$ for some ξ between x_{n+1} and x_n. Combining these, we need to show

$$\lim_{n\to\infty} \frac{F'(\xi)(x_{n+1} - x_n)}{x_{n+1} - x_n} = \lim_{n\to\infty} F'(\xi) = 0.$$

But the assumptions of the problem say $F'(x) = 0$ and we are told to assume F is continuously differentiable. But that says $\lim_{n\to\infty} F'(\xi) = 0$, as desired.

Problems 1.2, exercise 6: **a.** No. Informally this is because x_n has n^3 in it while $\alpha_n = n^2$. We want to show that there is *no* C so that $|x_n| \le C|\alpha_n|$. To show this we can take a limit:

$$\lim_{n\to\infty} \frac{|x_n|}{|\alpha_n|} = \lim_{n\to\infty} \frac{5n^2 + 9n^3 + 1}{n^2} = \lim_{n\to\infty} 5 + 9n + n^{-2} = +\infty$$

This means the fraction we took the limit of can be arbitrarily large, so there is no C for which

$$\frac{|x_n|}{|\alpha_n|} \le C.$$

b. No. *Similar.*

c. No. *Note*

$$\lim_{n\to\infty} |x_n|/|\alpha_n| = \lim_{n\to\infty} \sqrt{n+3} = +\infty.$$

d. Yes. Since

$$\lim_{n\to\infty} \frac{|x_n|}{|\alpha_n|} = \lim_{n\to\infty} \frac{5n^2 + 9n^3 + 1}{n^3} = \lim_{n\to\infty} 5n^{-1} + 9 + n^{-3} = 9,$$

any $C > 9$ works in the inequality

$$|x_n| \le C|\alpha_n|.$$

e. No. *Similar to* **c**.

Problems 1.2, exercise 10: **a.** Again we take a limit, but first we have to use Taylor's theorem with $f(x) = e^x$, $c = 0$, and $n = 1$:

$$e^x = 1 + x + 0.5e^{\xi}x^2$$

where ξ is a number between 0 and x, so $\lim_{x\to 0} \xi = 0$. Then

$$\lim_{x\to 0} \frac{|e^x - 1|}{|x^2|} = \lim_{x\to 0} \frac{|x + 0.5e^{\xi}x^2|}{x^2} \le \lim_{x\to 0} \left(|x|^{-1} + 0.5e^{\xi}\right) = +\infty + 0.5 = +\infty.$$

So "$e^x - 1 = O(x^2)$ as $x \to 0$" is not true because $|e^x - 1| \le C|x^2|$ implies

$$\frac{|e^x - 1|}{|x^2|} \le C.$$

b. To show that the "big-oh" is not true:

$$\lim_{x\to 0} \frac{|x^{-2}|}{|\cot x|} = \lim_{x\to 0} \frac{|\sin x|}{|x^2 \cos x|} = \lim_{x\to 0} \frac{|\sin x|}{|x|} \frac{1}{|\cos x|} \frac{1}{|x|} = \lim_{x\to 0} 1 \cdot 1 \cdot \frac{1}{|x|} = +\infty.$$

(You need, and may have memorized, that the limit of $(\sin x)/x$ as $x \to 0$ is one.) Thus "$x^{-2} = O(\cot x)$ as $x \to 0$" is not true.

c. We again take a limit:

$$\lim_{x\to 0} \frac{|\cot x|}{|x^{-1}|} = \lim_{x\to 0} \frac{|x||\cos x|}{|\sin x|} = 1.$$

If "$\cot x = o(x^{-1})$ as $x \to 0$" were true then this limit would be zero. (What we now know, however, is that

$$\cot x = O(x^{-1})$$

as $x \to 0$.)

Problems 3.1, exercise 2: **a.** Since $a_0 = 1.5$ and $b_0 = 3.5$, the initial interval has length $|b_0 - a_0| = 2$. The width at the nth step $|b_n - a_n| = 2^{-n}|b_0 - a_0| = 2^{-n+1}$.

b. By theorem 1, $|c_n - r| \le 2^{-(n+1)}|b_0 - a_0| = 2^{-n-1} \cdot 2 = 2^{-n}$. This says exactly that the maximum distance between the root and the midpoint of the nth interval is 2^{-n}.

Problems 3.1, exercise 7: The absolute error of the estimate from bisection, the nth midpoint c_n, is the distanece $|c_n - r|$. Thus by theorem 1 on p. 79, the question is: for what n is

$$2^{-(n+1)}|3-2| < 10^{-6} \quad ?$$

Trial and error, or using $\log_2$, quickly gets $n = 19$ as the first such integer n. The relative error is the ratio $|c_n - r|/|r|$. Note we do not know r but we do know $2 \leq r \leq 3$. Thus

$$\frac{|c_n - r|}{|r|} \leq \frac{|c_n - r|}{2} \leq \frac{2^{-(n+1)}|3-2|}{2} = 0.5 \cdot 2^{-(n+1)} = 2^{-(n+2)}$$

It follows that $n = 18$ is the first n for which we know that bisection will get relative accuracy of 10^{-6}.

Problems 3.1, exercise 1: *Not graded.*

Computer Problems 3.1, exercise 1ac: *Not graded.*