Exam # 2 SOLUTIONS.

See histogram of exam scores at http://www.cs.uaf.edu/~bueler/m310examhist.jpg

1 (a). We want to approximate the function \( f(x) = \sin(0.5x) \) by a second degree polynomial on the interval \([0, 3]\) using the values of \( f \) at \( x = 0, 2, 3 \) as interpolation points. Use Lagrange polynomials to find the polynomial. [No need to simplify the polynomial.]

**Solution.** The polynomial is \( p(x) = \sin(0)l_0(x) + \sin(1)l_1(x) + \sin(1.5)l_2(x) = \sin(1)l_1(x) + \sin(1.5)l_2(x) \), where
\[
l_1(x) = \frac{x(x-3)}{-2}, \quad l_2(x) = \frac{x(x-2)}{3}.
\]

(b). Estimate the maximum error in the above interpolation job. Use the interpolation errors theorem.

**Solution.** Here \( n = 2 \) in the interpolation error theorem I—note IET II does not apply since the points are not equally spaced:
\[
f(x) - p(x) = \frac{f'''(\xi)}{3!}(x - 0)(x - 2)(x - 3).
\]
And \( f'''(x) = -0.125 \sin(0.5x) \) which has maximum absolute value \( 0.125 \). Thus
\[
|f(x) - p(x)| \leq \frac{0.125}{6} |(x - 0)(x - 2)(x - 3)| \leq \frac{1}{6} \cdot \frac{1}{8} \cdot (3 \cdot 2 \cdot 3) = \frac{3}{8} = 0.375.
\]
*Less naive estimates of the maximum of \((x - 0)(x - 2)(x - 3)\) are appropriate but not essential.*

2 (a). Use Simpson’s rule to find the approximate value of \( \int_{-1}^{2} f(x) \, dx \) if
\[
\begin{array}{cccc}
x & -1 & 0.5 & 2 \\
f(x) & 1 & 1.1 & 3 \\
\end{array}
\]

**Solution.** \( \int_{-1}^{2} f(x) \, dx = \frac{15}{8} \cdot [1 + 4(1.1) + 3] = 4.2. \)

(b). In (a) we do not know anything about \( f(x) \) beyond its value at three points. What more information about \( f \) would you want? Explain how you would use that information to get an estimate of the accuracy of your result in (a).

**Solution.** One needs to know more values of \( f \) or a formula for \( f \) to compute better approximations. To know the accuracy of those approximations, one needs to know the fourth derivative or an upper bound \( M \) on the fourth derivative. If one knows \( M \) then an estimate of the error for the calculation in (a) is
\[
|E| \leq \frac{1}{90} M(1.5)^5.
\]

*Note: There was a misprint, which affected no one’s result, in the Simpson’s rule with error formula which was given on the exam. It should have said:*

Basic Simpson’s Rule with error. If \( h = \frac{b-a}{2} \) and \( c = \frac{a+b}{2} \) then
\[
\int_{a}^{b} f(x) \, dx = \frac{h}{3} [f(a) + 4f(c) + f(b)] - \frac{1}{90} h^5 f^{(4)}(\xi).]
\]

**Extra Credit.** Suppose we had the values of \( f \) at 33 equally spaced points, on the interval \([-1, 2]\), in (a). What algorithm would produce the best guess of the value of the integral? [Hint: 33 = 2^5 + 1.]

**Solution.** Romberg integration. Note that \( R(5, 0) \) would involve the calculation of trapezoid rule with 33 equally spaced points. Thus the first 6 rows of a Romberg table could be built, and \( R(5, 5) \) reported as the best estimate. Of course, we would not know much about the reliability of this estimate.
3. Suppose \( f \) is increasing and that we want to find \( \int_a^b f(x) \, dx \). Let \( P \) be a partition of the interval \([a, b]\) into \( n \) equal length subintervals. Show that the difference between the upper sum \( U(f; P) \) and the lower sum \( L(f; P) \) is

\[
\frac{b-a}{n} (f(b) - f(a)).
\]

**Solution.**

\[
U(f; P) - L(f; P) = \sum_{i=0}^{n-1} (M_i - m_i)(x_{i+1} - x_i) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_{i+1}) - f(x_i) = \frac{b-a}{n} [f(b) - f(a)],
\]

where \( x_i = a + \frac{b-a}{n} \) and, since \( f \) is increasing, \( M_i = f(x_{i+1}) \), \( m_i = f(x_i) \). The sum “telescopes,” of course.

**Extra Credit.** Produce a convincing and somewhat polished “Proof Without Words or Formulas” of 3 above. [*Hint: A picture with \( n = 5 \) and an “arbitrary” \( f(x) \) will be a good start.*]

**Solution.**

**4 (a).** Suppose you want a polynomial \( p(x) \) of degree 6 which is as accurate as possible as an approximation of \( e^x \) on the interval \([-0.5, 0.5]\). In particular, you want the coefficients of the polynomial. Explain how to do this in practice. You should explain with words and a pseudocode or Matlab. You may use built–in Matlab commands.

**Solution.** The obvious way to get the polynomial is by interpolation. Because we get to choose the interpolation points, we should use good points, that is, Chebyshev points. In detail this might mean

\[
N=6;
xx=.5*\cos((0:N)*\pi/N);
p=polyfit(xx,exp(xx),N);
\]

[I did not hold you responsible for the details of determining the Chebyshev points.]

**4 (b).** Suppose we want to compute \( e^x \) accurately for any \( x \in \mathbb{R} \). One method is to write \( x = n + r \) where \( n \) is an integer and \(-0.5 \leq r < 0.5\). Then we can reduce the calculation to a combination of multiplication and approximation of \( e^r \):

\[
e^x = e^{n + r} = (e^1)^n e^r \approx (e \cdot e \cdots e)p(r).
\]

(1)

**Complete** the following pseudocode by using (1) and including the polynomial \( p \) from (a): [*Note: It is not fair to use the exponential function in the above pseudocode.*]
function z = myexp(x);
% MYEXP uses polynomial interpolation to compute the exponential

n=floor(x+.5);
r=x-n;
e=2.718281828459045;

% the following is completion:
% assume p contains the coefficients of the polynomial from (a)

z=1;
for i=1:abs(n)
    z=z*e;
end
if n<0, z=1/z; end

z=z*polyval(p,r); % uses Horner’s evaluation of polynomial

[In fact, you should see myexp.m and also mycos.m on the course website. The strategy of this problem is very effective.]

5. Derive the midpoint rule with error formula for the integral
\[
\int_{-1}^{1} f(x) \, dx.
\]
[Hint: Use Taylor’s theorem. The error depends on the second derivative of \( f \).]

Solution. Starting with Taylor’s theorem around \( c = 0 \):
\[
f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2}x^2.
\]
Integrate to get
\[
\int_{-1}^{1} f(x) \, dx = 2f(0) + 0 + \frac{1}{2} \int_{-1}^{1} f''(\xi)x^2 \, dx.
\]
From the Mean Value Theorem for Integrals,
\[
\int_{-1}^{1} f''(\xi)x^2 \, dx = f''(\bar{\xi}) \int_{-1}^{1} x^2 \, dx = f''(\bar{\xi}) \frac{2}{3}
\]
for some \( \bar{\xi} \) in \([-1,1]\). Thus:
\[
\int_{-1}^{1} f(x) \, dx = 2f(0) + \frac{f''(\bar{\xi})}{3}.
\]

6. Determine \( n \) so that the composite trapezoid rule can be applied to the integral
\[
\int_{0}^{6} e^{-x^2/2} \, dx
\]
with maximum error \( 10^{-4} \).

Solution. Let \( h = 6/n \). We want \( n \) so that
\[
|E| = \frac{6}{12}h^2|f''(\xi)| = \frac{36}{2n^2}|f''(\xi)| < 10^{-4}.
\]
Note \( f''(x) = (x^2 - 1)e^{-x^2/2} \) so \( |f''(\xi)| \leq (36 - 1) \cdot 1 = 35. \)

[Actually \( |f''(\xi)| \leq 1 \), but you really have to work to see this.]
We need
\[
\frac{36 \cdot 35}{2n^2} = \frac{18 \cdot 35}{n^2} < 10^{-4}
\]
or
\[ n > 10^2 \sqrt{18} \sqrt{35} \approx 2510. \]

7. Use Taylor’s theorem to derive the error term \( E \) in the famous finite difference approximation for the second derivative:
\[
f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + E.
\]

**Solution.** Write out the Taylor’s theorem for \( f(x + h) \) and \( f(x - h) \). Noting that the odd terms will cancel when we add, and that the second derivative is what we want, we use \( n = 3 \):
\[
\begin{align*}
f(x + h) &= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_1)}{4!}h^4, \\
f(x - h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_2)}{4!}h^4.
\end{align*}
\]
Adding these, and moving the “\(2f(x)\)” term to the left,
\[
f(x + h) - 2f(x) + f(x - h) = h^2 f''(x) + \frac{h^4}{4!} \left[ f^{(4)}(\xi_1) + f^{(4)}(\xi_2) \right].
\]
Dividing by \( h^2 \) and noting that we want an average of fourth derivatives:
\[
\frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x) + \frac{h^2}{12} \left[ f^{(4)}(\xi_1) + f^{(4)}(\xi_2) \right].
\]
By the Intermediate Value Theorem, and assuming that the fourth derivative is continuous, and rearranging into the desired form:
\[
f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi).
\]