Exam # 2 *SOLUTIONS.*

See histogram of exam scores at `http://www.cs.uaf.edu/~bueler/m310examhist.jpg`

1 (a). We want to approximate the function $f(x) = \sin(.5x)$ by a second degree polynomial on the interval $[0,3]$ using the values of f at $x = 0, 2, 3$ as interpolation points. Use Lagrange polynomials to find the polynomial. [No need to simplify the polynomial.]

Solution. The polynomial is $p(x) = \sin(0)l_0(x) + \sin(1)l_1(x) + \sin(1.5)l_2(x) = \sin(1)l_1(x) + \sin(1.5)l_2(x)$, where

$$l_1(x) = \frac{x(x-3)}{-2}, \quad l_2(x) = \frac{x(x-2)}{3}.$$

(b). Estimate the maximum error in the above interpolation job. Use the interpolation errors theorem.

Solution. Here $n = 2$ in the interpolation error theorem I—*note IET II does not apply since the points are not equally spaced*:

$$f(x) - p(x) = \frac{f'''(\xi)}{3!}(x-0)(x-2)(x-3).$$

And $f'''(x) = -.125\sin(.5x)$ which has maximum absolute value .125. Thus

$$|f(x) - p(x)| \le \frac{.125}{6}|(x-0)(x-2)(x-3)| \le \frac{1}{6 \cdot 8}(3 \cdot 2 \cdot 3) = \frac{3}{8} = .375.$$

[*Less naive estimates of the maximum of* $(x-0)(x-2)(x-3)$ *are appropriate but not essential.*]

2 (a). Use Simpson's rule to find the approximate value of $\int_{-1}^{2} f(x)\,dx$ if

x	-1	$.5$	2
$f(x)$	1	1.1	3

Solution. $\int_{-1}^{2} f(x)\,dx = \frac{1.5}{3}\left[1 + 4(1.1) + 3\right] = 4.2.$

(b). In **(a)** we do not know anything about $f(x)$ beyond its value at three points. What more information about f would you want? Explain how you would use that information to get an estimate of the accuracy of your result in **(a)**.

Solution. One needs to know more values of f or a formula for f to compute better approximations. To know the accuracy of those approximations, one needs to know the fourth derivative or an upper bound M on the fourth derivative. If one knows M then an estimate of the error for the calculation in **(a)** is

$$|E| \le \frac{1}{90}M(1.5)^5.$$

[*Note*: There was a misprint, which affected no ones' result, in the Simpson's rule with error formula which was given on the exam. It should have said:

Basic Simpson's Rule with error. If $h = \frac{b-a}{2}$ and $c = \frac{a+b}{2}$ then

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right] - \frac{1}{90}h^5 f^{(4)}(\xi).]$$

Extra Credit. Suppose we had the values of f at 33 equally spaced points, on the interval $[-1,2]$, in **(a)**. What algorithm would produce the best guess of the value of the integral? [*Hint*: $33 = 2^5 + 1$.]

Solution. Romberg integration. Note that $R(5,0)$ would involve the calculation of trapezoid rule with 33 equally spaced points. Thus the first 6 rows of a Romberg table could be built, and $R(5,5)$ reported as the best estimate. Of course, we would not know much about the reliability of this estimate.

3. Suppose f is increasing and that we want to find $\int_a^b f(x)\,dx$. Let P be a partition of the interval $[a,b]$ into n equal length subintervals. Show that the difference between the upper sum $U(f;P)$ and the lower sum $L(f;P)$ is

$$\frac{b-a}{n}\left(f(b)-f(a)\right).$$

Solution.

$$U(f;P)-L(f;P)=\sum_{i=0}^{n-1}(M_i-m_i)(x_{i+1}-x_i)=\frac{b-a}{n}\sum_{i=0}^{n-1}f(x_{i+1})-f(x_i)=\frac{b-a}{n}[f(b)-f(a)],$$

where $x_i=a+i\frac{b-a}{n}$ and, since f is increasing, $M_i=f(x_{i+1})$, $m_i=f(x_i)$. The sum "telescopes," of course.

Extra Credit. Produce a convincing and somewhat polished "*Proof Without Words or Formulas*" of **3** above. [*Hint*: A picture with $n=5$ and an "arbitrary" $f(x)$ will be a good start.]

Solution.

4 (a). Suppose you want a polynomial $p(x)$ of degree 6 which is as accurate as possible as an approximation of e^x on the interval $[-.5,.5]$. In particular, you want the coefficients of the polynomial. Explain *how* to do this in practice. You should explain with words and a pseudocode or *Matlab*. You may use built–in *Matlab* commands.

Solution. The obvious way to get the polynomial is by interpolation. Because we get to *choose* the interpolation points, we should use good points, that is, Chebyshev points. In detail this might mean

```
N=6;
xx=.5*cos((0:N)*pi/N);
p=polyfit(xx,exp(xx),N);
```

[*I did not hold you responsible for the details of determining the Chebyshev points.*]

(b). Suppose we want to compute e^x accurately for any $x\in\mathbb{R}$. One method is to write $x=n+r$ where n is an integer and $-.5\le r<.5$. Then we can reduce the calculation to a combination of multiplication and approximation of e^r:

$$e^x=e^{n+r}=(e^1)^n\,e^r\approx(e\cdot e\cdots e)p(r). \tag{1}$$

Complete the following pseudocode by using (1) and including the polynomial p from **(a)**: [*Note*: It is not fair to *use* the exponential function in the above pseudocode.]

```
function z = myexp(x);
% MYEXP uses polynomial interpolation to compute the exponential

n=floor(x+.5);
r=x-n;
e=2.718281828459045;

% the following is completion:

% assume  p  contains the coefficients of the polynomial from (a)

z=1;
for i=1:abs(n)
   z=z*e;
end
if n<0, z=1/z; end

z=z*polyval(p,r);  % uses Horner's evaluation of polynomial
```

[*In fact*, you should see `myexp.m` and also `mycos.m` on the course website. The strategy of this problem is very effective.]

5. Derive the midpoint rule with error formula for the integral

$$\int_{-1}^{1} f(x)\,dx.$$

[*Hint*: Use Taylor's theorem. The error depends on the second derivative of f.]

Solution. Starting with Taylor's theorem around $c=0$: $f(x)=f(0)+f'(0)x+\frac{f''(\xi)}{2}x^2$. Integrate to get

$$\int_{-1}^{1} f(x)\,dx = 2f(0)+0+\frac{1}{2}\int_{-1}^{1} f''(\xi)x^2\,dx.$$

From the Mean Value Theorem for Integrals,

$$\int_{-1}^{1} f''(\xi)x^2\,dx = f''(\bar\xi)\int_{-1}^{1} x^2\,dx = f''(\bar\xi)\frac{2}{3}$$

for some $\bar\xi$ in $[-1,1]$. Thus:

$$\int_{-1}^{1} f(x)\,dx = 2f(0)+\frac{f''(\bar\xi)}{3}.$$

6. Determine n so that the composite trapezoid rule can be applied to the integral

$$\int_{0}^{6} e^{-x^2/2}\,dx$$

with maximum error 10^{-4}.

Solution. Let $h=6/n$. We want n so that

$$|E| = \frac{6}{12}h^2|f''(\xi)| = \frac{36}{2n^2}|f''(\xi)| < 10^{-4}.$$

Note $f''(x)=(x^2-1)e^{-x^2/2}$ so

$$|f''(\xi)| \le (36-1)\cdot 1 = 35.$$

[*Actually* $|f''(\xi)|\le 1$, but you really have to work to see this.]

We need

$$\frac{36\cdot 35}{2n^2} = \frac{18\cdot 35}{n^2} < 10^{-4}$$

or

$$n > 10^2\sqrt{18}\sqrt{35} \approx 2510.$$

7. Use Taylor's theorem to derive the error term E in the famous finite difference approximation for the second derivative:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + E.$$

Solution. Write out the Taylor's theorem for $f(x+h)$ and $f(x-h)$. Noting that the odd terms will cancel when we add, and that the second derivative is what we want, we use $n = 3$:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_1)}{4!}h^4,$$
$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_2)}{4!}h^4.$$

Adding these, and moving the "$2f(x)$" term to the left,

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(x) + \frac{h^4}{4!}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right].$$

Dividing by h^2 and noting that we want an average of fourth derivatives:

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{h^2}{12}\left[\frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{2}\right].$$

By the Intermediate Value Theorem, and assuming that the fourth derivative is continuous, and rearranging into the desired form:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{h^2}{12}f^{(4)}(\xi).$$