Selected Assignment # 7 Solutions.

[I graded 5.3 #4, 5.3 #7, 5.3 CP #4, 5.4 #2, Problem A, 5.5 #2, 5.5 #6, 5.5 CP #1 and 5.2 CP #4. Each was worth 5 points for a total of 45.]

5.3 #4. Let $h = b - a = 4$. Then apply the trapezoid rule with extrapolation (that is, Romberg!) to the integral $\int_0^4 2^x\,dx$:

$$R(1,0) = \frac{h}{4}[2^0 + 2\cdot 2^2 + 2^4] = 25;$$

$$R(2,0) = \frac{h}{8}[2^0 + 2\cdot 2^1 + 2\cdot 2^2 + 2\cdot 2^3 + 2^4] = 22.5;$$

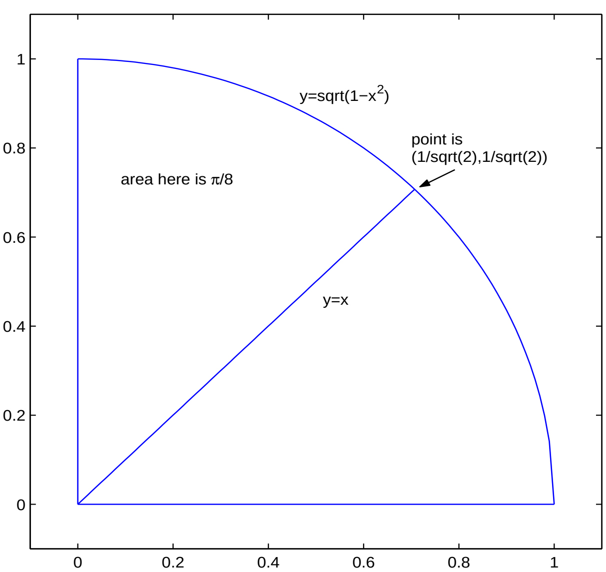
$$R(2,1) = R(2,0) + \frac{1}{3}\left[R(2,0) - R(1,0)\right] = 22.5 + \frac{1}{3}(22.5 - 25) = 21.6667.$$

Compare to the correct answer

$$\int_0^4 2^x\,dx = \left.\frac{2^x}{\ln 2}\right]_0^4 = 21.6404.$$

5.3 #7. The method may work, but it will converge slowly to the correct answer. Note that, like $g(x) = \sqrt{x}$, the integrand $f(x) = \sqrt{x}\cos x$ has no derivative (or higher derivatives) at $x = 0$, which is one end of the interval. For Romberg to work we want derivatives of high order to be of reasonable size. In particular see the Euler–Maclaurin formula with error term.

5.3 CP #4. Your diagram (it can be by hand, of course!) should look like:



We now do the integral by using `romberg`, which was handed out on paper in class. First define `f` in an m–file:

```
function y=f(x);
y=sqrt(1-x.^2)-x;
```

Now integrate

```
>> q=romberg(0,1/sqrt(2),1e-12)
          q             cnt     error est
  3.921430153752e-001     3    3.859e-002
  3.926820549563e-001     5    5.390e-004
  3.926987182501e-001     9    1.666e-005
  3.926990777811e-001    17    3.595e-007
  3.926990816805e-001    33    3.899e-009
  3.926990816987e-001    65    1.816e-011
  3.926990816987e-001   129    3.342e-014
q =
         0.392699081698724
>> [8*q pi]'
ans =
          3.14159265358979
          3.14159265358979
```

Note that the function was only evaluate 129 times to determine π to 15 digits! Romberg rocks!

5.4 #2. [*I only graded* **2 a.** *and* **2 b.** *because the 3/8 rule was not well documented in the book. The composite Simpson's rule with error term is found at the top of page 228.*]
a. Here $f(x) = \sin(\pi x^2/2)$ so $f''(x) = \pi\cos(\pi x^2/2) - \pi^2 x^2 \sin(\pi x^2/2)$. The error term in the trapezoid rule is

$$E_T = -\frac{b-a}{12} h^2 f''(\xi).$$

We want to choose h so that the error term is at most 10^{-3}: $\left|\frac{1}{12} h^2 f''(\xi)\right| \le 10^{-3}$. But

$$|f''(\xi)| \le \pi \cdot 1 + \pi^2 \cdot 1^2 \cdot 1 = \pi(1+\pi).$$

[*We have crucially used the fact that ξ is in the interval $\xi \in [0,1]$.*] Thus

$$\frac{1}{12} h^2 \pi(1+\pi) \le 10^{-3} \qquad \Longleftrightarrow \qquad h \le \sqrt{\frac{12}{13.01 \cdot 10^3}} = 0.0304.$$

This corresponds to $n \ge 33$.
b. Here we use the error term for the composite Simpson's rule on page 228:

$$E_S = -\frac{b-a}{180} h^4 f^{(4)}(\xi).$$

Note $f^{(4)}(x) = \pi^4 x^4 \sin(\pi x^2/2) - 6\pi^3 x^2 \cos(\pi x^2/2) - 3\pi^2 \sin(\pi x^2/2)$ so $|f^{(4)}(\xi)| \le \pi^4 + 6\pi^3 + 3\pi^2 \approx 313.1$. [*I have replaced all the x's with 1 and the* sin/cos*'s with 1, and all the minus' with plus' because I can't count on cancellation. The interval $x \in [0,1]$ matters!*]

Thus we want h small enough so that

$$|E_S| = \frac{1}{180} h^4 \left|f^{(4)}(\xi)\right| \le \frac{h^4}{180} 313.1 \le 10^{-3}$$

which is equivalent to $h \le \sqrt[4]{\frac{180}{313.1 \cdot 10^3}} \approx 0.1548$. This corresponds to $n \ge 7$ so Simpson's is quite an improvement on the trapezoid rule for this problem.

Problem A. [*As with* **2**, *I only graded the trapezoid and Simpson's rule and not the Simpson's 3/8 part.*]
a. Here our upper bound on $f''(\xi)$ is different because ξ is in the interval $[0,4]$:

$$|f''(\xi)| \le \pi + \pi^2 4^2 \approx 161.1.$$

Thus

$$\frac{b-a}{12}h^2|f''(\xi)| \le \frac{h^2}{3}161.1 \le 10^{-3} \qquad \Longleftrightarrow \qquad h \le \sqrt{\frac{3}{161.1\cdot 10^3}} \approx .00432$$

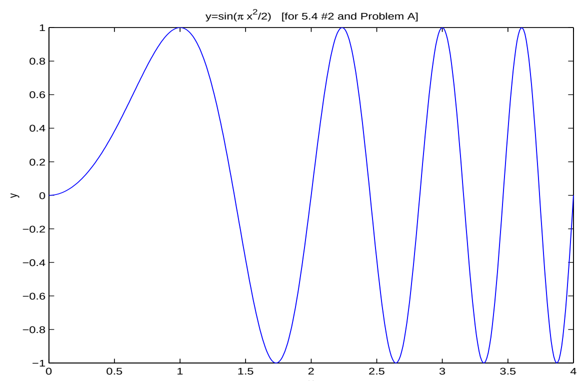
which corresponds to n about $n \ge 927$. Note this is *a lot* more than four times the n in **2 a**. The plot below makes it very clear why: the derivatives are all increasing rapidly as we move to the right on the graph of f.

b. Here $|f^{(4)}(\xi)| \le 4^4\pi^4 + 6\cdot 4^2\pi^3 + 3\pi^2 \approx 27943$ for $\xi \in [0,4]$ so

$$|E_S| \le \frac{4}{180}h^4 27943 \le 10^{-3} \qquad \Longleftrightarrow \qquad h \le \sqrt[4]{\frac{45}{10^3\cdot 27943}} \approx 0.0356.$$

Notice that this h is much smaller (almost five times) than the h in **2 b.** *The reason is purely that the interval has expanded to include places where the fourth derivative of f is much larger. This example shows why we want to* adapt.

Here is the figure requested in the statement of the problem:



5.4 CP #5.

[*NOT GRADED!*] For the input

```
>> f=inline('cos(2*x)./exp(x)','x')
f =
     Inline function:
     f(x) = cos(2*x)./exp(x)
>> quad(f,0,2*pi,1e-3,1)
ans =
          0.199626136373597
```

I get the picture:

Note that the figure shows the points at which `quad` evaluated the function. Note the adaptivity. Note the use of a low tolerance of 10^{-3}, which can obviously be improved at the cost of more evaluations.

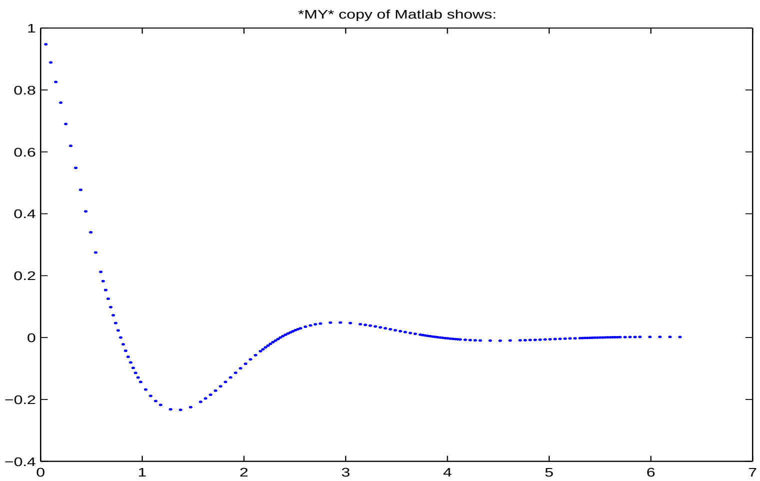
[*Unfortunately I think this picture must be produced in a slightly different manner in version 6 of* Matlab *(in the Chapman 103 lab) works. One ought to be able to produce a comparable picture with* ***quad8****.*]

5.5 #2.

[*I graded only parts* **a** *and* **b**.] Here is a problem which rewards *Matlab* usage:

a.

```
>> x=[-sqrt(1/3) sqrt(1/3)]'; A=[1 1];
>> A*[1 1]'
ans =
     2
>> A*x
```




```
ans =
     0
>> A*(x.^2)
ans =
         0.666666666666667
>> A*(x.^3)
ans =
     0
>> A*(x.^4)
ans =
         0.222222222222222
```

The above computations correspond to:

$$\int_{-1}^{1} 1\,dx = 2; \qquad \int_{-1}^{1} x\,dx = 0; \qquad \int_{-1}^{1} x^2\,dx = 2/3; \qquad \int_{-1}^{1} x^3\,dx = 0; \qquad \int_{-1}^{1} x^4\,dx = 2/5.$$

I have included the last equation because it shows that the $n = 1$ Gauss rule is only correct up to degree $2n + 1 = 3$ and is not correct for degree 4.

b.

```
>> x=[-sqrt((1/7)*(3+4*sqrt(.3))) -sqrt((1/7)*(3-4*sqrt(.3)))...
sqrt((1/7)*(3-4*sqrt(.3))) sqrt((1/7)*(3+4*sqrt(.3)))]'
x =
        -0.861136311594053
        -0.339981043584856
         0.339981043584856
         0.861136311594053
>> A=[.5-sqrt(10/3)/12 .5+sqrt(10/3)/12 .5+sqrt(10/3)/12 .5-sqrt(10/3)/12]
A =
  Columns 1 through 3
         0.347854845137454         0.652145154862546         0.652145154862546
  Column 4
         0.347854845137454
>> [A*[1 1 1 1]' A*(x.^1) A*(x.^2) A*(x.^3) A*(x.^4) A*(x.^5) A*(x.^6) A*(x.^7)]
ans =
  Columns 1 through 3
```

```
                          2    -5.55111512312578e-017         0.666666666666667
  Columns 4 through 6
                          0                       0.4                         0
  Columns 7 through 8
         0.285714285714286                          0
>> [2 0 2/3 0 2/5 0 2/7 0]
ans =
  Columns 1 through 3
                          2                         0         0.666666666666667
  Columns 4 through 6
                          0                       0.4                         0
  Columns 7 through 8
         0.285714285714286                          0
```

c. Similar.

5.5 #6. We only get up to x^1, unfortunately:

$$b-a=\int_a^b 1\,dx = A\cdot 1 + B\cdot 0;$$

$$\frac{b^2-a^2}{2}=\int_a^b x\,dx = A\cdot a + B\cdot 1.$$

These two already determine A and B:

$$A=b-a;$$

$$B=\frac{b^2-a^2}{2}-aA=\frac{b^2-a^2-2ab+2a^2}{2}=\frac{(b-a)^2}{2}.$$

In conclusion the rule is only accurate for degree one polynomials.

5.5 CP #1. Based on the $n=2$ part of the table on page 235, and on formula (6) on the top of page 233, I wrote the following program:

```
function q=mygauss2(f,a,b);
% MYGAUSS2 integrates f by the n=2 Gauss rule
%
% f might be ">> f=inline('exp(-cos(x).^2)','x') "
% then ">> mygauss2(f,0,2) "

t=[-sqrt(3/5) 0 sqrt(3/5)]';

A=[5 8 5]/9;

c=(b-a)/2; d=(b+a)/2;

q=c*A*feval(f,c*t+d);
```

5.5 CP #2. On the integral $\int_0^2 e^{\cos^2 x}\,dx$ it produces:

```
>> f=inline('exp(-cos(x).^2)','x')
f =
     Inline function:
     f(x) = exp(-cos(x).^2)
>> mygauss2(f,0,2)
ans =
          1.41194709649912
```

versus:

```
>> quad8(f,0,2)
ans =
          1.41830206582229
>> quad(f,0,2,1e-8)
ans =
          1.41830206579667
```

On the integral $\int_0^1 dx/\sqrt{x}$ it produces $1.7508\ldots$ versus the correct answer of 2. The error is not too surprising because the unbounded function $x^{-1/2}$ is poorly approximated by a polynomial.