Selected Assignment # 6 Solutions.

[I graded 5.1 #2, 5.1 #4, 5.1 #6, 5.2 #4, 5.2 #9, 5.2 #19, and 5.2 CP #4. Each was worth 5 point for a total of 35.]

5.1 #2.

$$\int_1^2 \frac{1}{x}\,dx \le U(\frac{1}{x};P) = \left(\frac{1}{1}\right)\cdot\frac{1}{2} + \left(\frac{1}{3/2}\right)\cdot\frac{1}{2} = \frac{5}{6} = .8333\ldots.$$

Compare to $\ln(2) = 0.693147\ldots$.

5.1 #4. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of n uniform subintervals. Note $x_{i+1} - x_i = \frac{b-a}{n}$. Let f be a decreasing function over $[a,b]$. Then

$$U(f;P) - L(f;P) = \sum_{i=0}^{n-1} M_i(x_{i+1}-x_i) - m_i(x_{i+1}-x_i) = \frac{b-a}{n}\sum_{i=0}^{n-1} M_i - m_i$$

where $M_i = \max_{x\in[x_i,x_{i+1}]} f(x) = f(x_i)$ since f is decreasing. Similarly $m_i = \min_{x\in[x_i,x_{i+1}]} f(x) = f(x_{i+1})3$.

Thus we get a telescoping sum:

$$\begin{aligned}
U(f;P) - L(f;P) &= \frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i) - f(x_{i+1}) \\
&= \frac{b-a}{n}\left(f(x_0) - f(x_1) + f(x_1) - f(x_2) + \cdots + f(x_{n-1}) - f(x_n)\right) \\
&= \frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(a) - f(b)).
\end{aligned}$$

5.1 #6. [*I interpret "*log*" to mean* $\log_e = \ln$.] Here $f(x) = (\log(x))^{-1}$. From **# 4**, since f is decreasing,

$$U(f;P) - L(f;P) = \frac{b-a}{n}\left(f(a) - f(b)\right) = \frac{3}{n}\left(\frac{1}{\log 2} - \frac{1}{\log 5}\right).$$

Since the correct integral is between $L(f;P)$ and $U(f;P)$ we choose $\frac{1}{2}(L+U)$ as the estimate. We need to choose n so that the above quantity is less than $1/2 \times 10^{-4} = .00005$:

$$\frac{1}{2}\frac{3}{n}\left(\frac{1}{\log 2} - \frac{1}{\log 5}\right) \le \frac{1}{2}\times 10^{-4} \qquad \text{or} \qquad n \ge 3\times 10^4 \times \left(\frac{1}{\log 2} - \frac{1}{\log 5}\right) \approx 24640.8$$

so choosing $n = 24641$ will suffice.

5.2 #4. In this case the error formula is

$$-\frac{6-0}{12}(6/100)^2 f''(\zeta)$$

where $f(x) = \sin(x^2)$. So $f''(x) = 2\cos(x^2) - (2x)^2\sin(x^2)$ and

$$|f''(x)| \le 2\cdot 1 + 4\cdot 6^2\cdot 1 = 146 = M.$$

Thus with $n = 100$ subintervals

$$\left|\int_0^6 \sin(x^2)\,dx - T\right| \le \frac{6\cdot 36}{12\cdot 10000}(146) \approx 0.26.$$

This is not very good. A look at the graph of $\sin(x^2)$ on the interval $x \in [0,6]$ will show why the trapezoid rule with only 100 subintervals is far from perfect.

5.2 #9. If f is concave downward (and if f'' exists) then $f'' \le 0$. But then the error formula for the trapezoid rule says

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12}h^2 f''(\xi) \ge T - 0 = T,$$

that is, the trapezoid rule underestimates the integral. (*The fact you are supposed to prove* is *geometrically obvious, and in fact the true definition of concave down, instead of relating to the second derivative, is essentially the statement that secant lines are below the graph of the function. See any good calculus book.*)

5.2 #19. *Context.* We want a rule so that for $x_0, x_1, \ldots, x_n$ fixed we can approximate

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^n w_i f(x_i) \tag{1}$$

and at least know that this rule is exact if $f(x)$ is a polynomial $p(x)$ of degree at most n. It is not supposed to be instantly obvious that this is possible—we see it is true as follows:

Proof. Given $x_0, x_1, \ldots, x_n$, the unique polynomial $p(x)$ which goes through the corresponding values $f(x_0), f(x_1), \ldots, f(x_n)$ is given by $p(x) = \sum_{i=0}^n f(x_i) l_i(x)$ where $l_i(x) = \prod_{j=0\,(i\ne j)}^n \frac{x-x_j}{x_i-x_j}$. If (**??**) is to be true for *all* polynomials then it is true for the polynomials $l_i(x)$ in particular. Thus we want:

$$\int_a^b l_i(x)\,dx = \sum_{j=0}^n w_j l_i(x_j) = \sum_{j=0}^n w_j \delta_{ij} = w_i$$

where $\delta_{ij} = 1$ if $i = j$ and is zero otherwise. [*Remember what properties the polynomials $l_i(x)$ have!*] That is, $w_i = \int_a^b l_i(x)\,dx$, as desired.

5.2 CP #4. *You have choices on these problems, and a really bad choice will lose a point even if you did the bad choice well. For instance, a single upper sum $\int_0^{0.8} \frac{\sin x}{x}\,dx \approx 1 \cdot 0.8 = 0.8$ is not very impressive, though it is "an approximation".*

I chose to use the trapezoid rule with n equal length subintervals, and to choose n sufficiently large so that the error is at most 10^{-6}. By the error formula for the composite trapezoid rule I want:

$$\frac{b-a}{12}\left(\frac{b-a}{n}\right)^2 M = \frac{1}{15}\left(\frac{4}{5n}\right)^2 M = \frac{16M}{375n^2} \le 10^{-6}$$

where M is an upper bound on $|f''(x)|$ on the interval $x \in [0, 0.8]$.

Here $f(x) = \sin(x)/x$, of course, and we should make it continuous by choosing $f(0) = 0$. I calculated that

$$f''(x) = -\frac{\sin x}{x} + 2\frac{1-\cos x}{x^2} + 2\frac{\sin x - x}{x^3},$$

which is bounded. In fact, by graphing or thinking we see that $M = 1/3$.

Then $16M/375n^2 < 10^{-6}$ if $n > 119.2$ so I chose $n = 120$ in trapezoid rule and got

$$\int_0^{0.8} \frac{\sin x}{x}\,dx \approx 0.7720948$$

which compares to 0.77209578548 from both Simpson's rule with $n = 120$ subintervals and the $n = 8$ case of Romberg integration. Very probably we have achieved our 10^{-6} error. [*The mentioned figures from trapezoid, Simpson's and Romberg all require comparable amounts of work, and do better than trapezoid. Note that you can also get this answer by hand in about 15 minutes if you use the Taylor expansion of $f(x)$.*]