Selected Assignment # 5 Solutions.

[I graded 4.2 #5, 4.2 #6, 4.2 CP #7, 4.3 #6, 4.3 #8, 1.1 #2. Each was worth 5 point for a total of 30.]

4.2 #5. The error in linear interpolation is given by the $n = 1$ form of the interpolation errors theorem i:

$$f(x) - p(x) = \frac{1}{2} f''(\xi)(x - x_0)(x - x_1)$$

where $p(x)$ is the linear function through the two points.

We are *interpolating*, so

$$|f(x) - p(x)| \le \frac{1}{2} M |(x - x_0)(x - x_1)|$$

where M is defined in the problem statement, and we seek the maximum of the function

$$g(x) = |(x - x_0)(x - x_1)|$$

on the interval $x_0 \le x \le x_1$. But this is easy to find by calculus, for instance:

$$\max_{x_0 \le x \le x_1} g(x) = \frac{1}{4}(x_1 - x_0)^2.$$

Thus we get $|f(x) - p(x)| \le \frac{1}{8} M (x_1 - x_0)^2$. [This problem can also be done directly from the interpolation errors theorem II.]

4.2 #6. From the above problem,

$$|\sin(x) - p(x)| \le \frac{1}{8} M (x_1 - x_0)^2 \le \frac{1}{8}(\max|\sin(x)|)(0.1)^2 = 1.25 \times 10^{-5}.$$

Since the entries in the table are correct to ten decimal places, almost all the error will come from the linear interpolation, and we expect at most 5 correct decimal places.

4.2 CP #7. *[I interpret this problem as calling for a plot not a long list of numbers.]*

The code I wrote follows, along with the output figures. I found that the plots were too big to print well if I showed all 5 cases here, so to save paper I just did the $n = 4, 8, 16$ cases.

```
% problem CP #7 in section 4.2

f=inline('(1-x).*((1-x)>0)','x');
xx=linspace(-4,4,128);

figure(1)
for i=1:3
   n=2^(i+1); x=linspace(-4,4,n+1); y=f(x);
   p=polyfit(x,y,n);
   subplot(3,2,2*i-1)
   plot(x,y,'o',xx,f(xx),':',xx,polyval(p,xx))
   text(-3,0,['n = ' num2str(n) ' degree'])
   if i==1, legend('interpolation pts','f','polynomial')
      title('equally spaced interpolation points'), end
   subplot(3,2,2*i)
   plot(xx,abs(f(xx)-polyval(p,xx)))
   if i==1, legend('|f(x)-p(x)|'), end
```

```
end

figure(2)
for i=1:3
   n=2^(i+1);
   j=0:n; xj=4*cos(j*pi/n); yj=f(xj);
   pj=polyfit(xj,yj,n);
   subplot(3,2,2*i-1)
   plot(xj,yj,'o',xx,f(xx),':',xx,polyval(pj,xx))
   text(-3,0,['n = ' num2str(n) ' degree'])
   if i==1, legend('interpolation pts','f','polynomial')
      title('Chebyshev interpolation points'), end
   subplot(3,2,2*i)
   plot(xx,abs(f(xx)-polyval(pj,xx)))
   if i==1, legend('|f(x)-p(x)|'), end
end
```

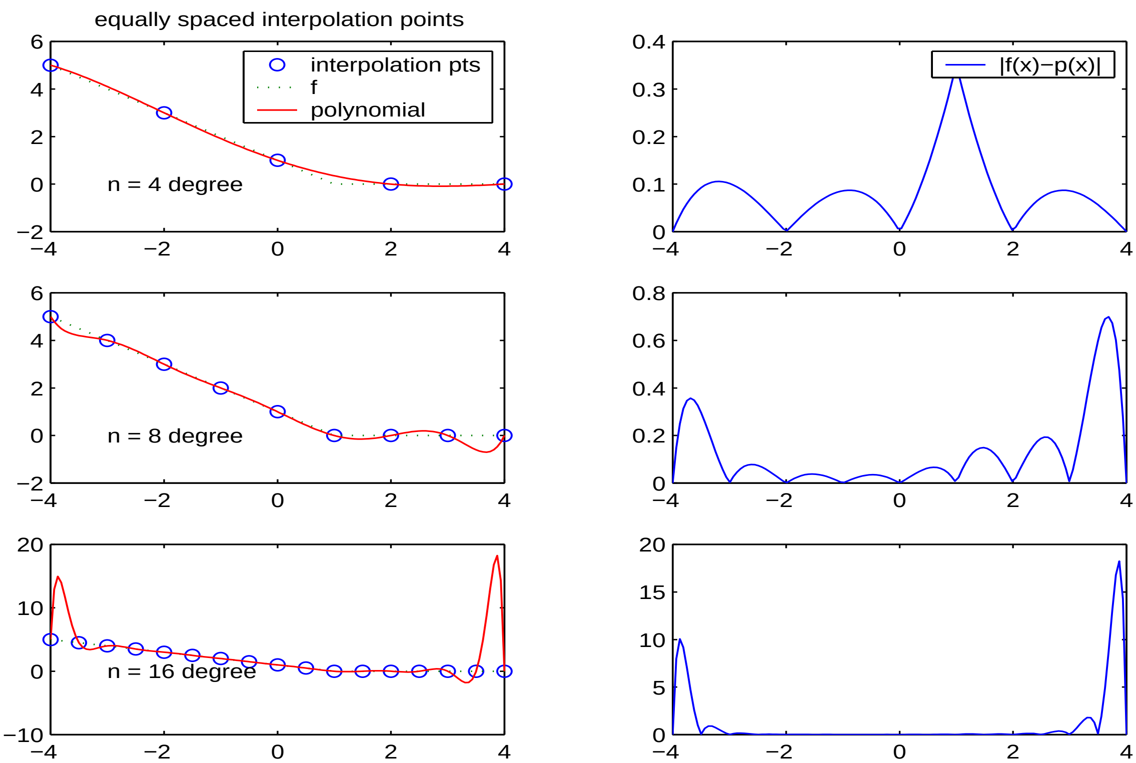

FIGURE 1. For **4.2 CP # 7.**

4.3 #6. The problem is that in the expressions for $f(x+h) - f(x)$ and $f(x-h) - f(x)$ there must be different values of ξ since $x+h$ and $x-h$ are different points. Note that the analysis is correct if the function $f'''(x)$ is constant, that is, for $f(x)$ a cubic polynomial.

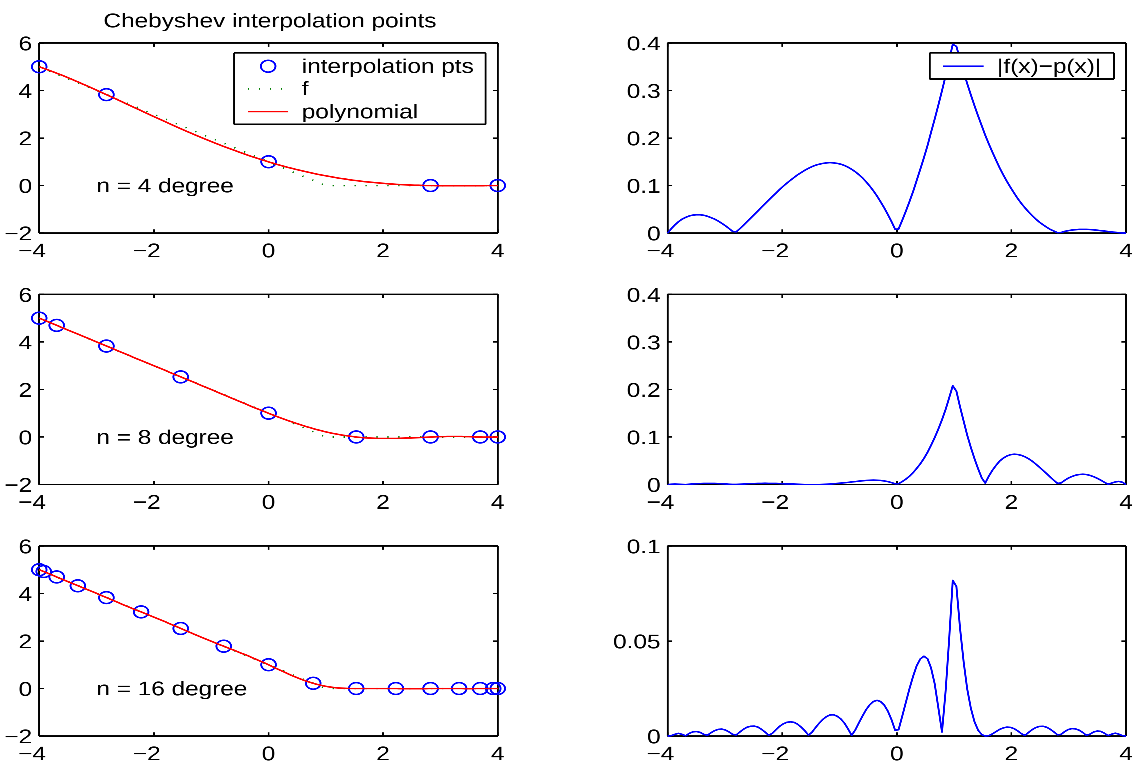

FIGURE 2. For **4.2 CP # 7.**

4.3 #8. **a.** This is similar to what is on page 173. Use $n = 2$ form of Taylor's Theorem:

$$\frac{1}{4h}\left[f(x+2h) - f(x-2h)\right]$$
$$= \frac{1}{4h}\left[f(x) + f'(x)2h + \frac{f''(x)}{2}(2h)^2 + \frac{f'''(\xi_1)}{3!}(2h)^3 - \left(f(x) - f'(x)2h + \frac{f''(x)}{2}(2h)^2 - \frac{f'''(\xi_2)}{3!}(2h)^3\right)\right]$$
$$= f'(x) + \frac{(2h)^3}{3!(4h)}\left(f'''(\xi_1) + f'''(\xi_2)\right) = f'(x) + \frac{2}{3}h^2\left(\frac{f'''(\xi_1) + f'''(\xi_2)}{2}\right) = f'(x) + \frac{2}{3}h^2 f'''(\xi),$$

so

$$f'(x) = \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] - \frac{2}{3}h^2 f'''(\xi).$$

b. This is similar to the calculation on page 180. From the $n = 3$ form of Taylor's theorem:

$$f(x+2h) = f(x) + f'(x)2h + \frac{f''(x)}{2}(2h)^2 + \frac{f'''(x)}{3!}(2h)^3 + \frac{f^{(4)}(\xi_1)}{4!}(2h)^4,$$
$$f(x-2h) = f(x) - f'(x)2h + \frac{f''(x)}{2}(2h)^2 - \frac{f'''(x)}{3!}(2h)^3 + \frac{f^{(4)}(\xi_2)}{4!}(2h)^4.$$

By adding these and removing the "$2f(x)$" term:

$$\frac{1}{4h^2}\left[f(x+2h) - 2f(x) + f(x-2h)\right] = \frac{1}{4h^2}\left[f''(x)4h^2 + \frac{2}{3}h^4\left(f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right)\right]$$
$$= f''(x) + \frac{1}{3}h^2\left(\frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{2}\right).$$

It follows that

$$f''(x) = \frac{1}{4h^2}\left[f(x+2h) - 2f(x) + f(x-2h)\right] - \frac{1}{3}h^2 f^{(4)}(\xi).$$

1.1 #2. I chose $f(x) = \cosh(x)$.

```
% problem CP #2 in section 1.1

j=1:30; h=4.^(-(1:30)); x=1;
d=(cosh(x+h)-cosh(x))./h;
err=abs(d-sinh(x));
semilogy(j,err,'o')
xlabel('j axis (h=4^-j)'), ylabel('absolute error')
title('error in differentiating cosh(x)')
```

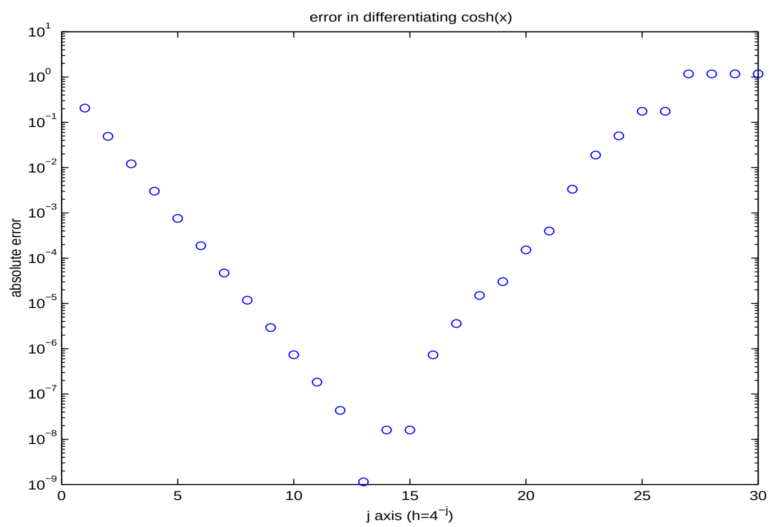

FIGURE 3. For **1.1 CP # 2.**

[*Interpret the results!*] Note that as j increases from 1 to 13 the error decreases to a minimum of 10^{-9} or so. This follows from the increasing accuracy of $\frac{f(x+h)-f(x)}{h}$ as an approximation of $f'(x)$. That is, the *truncation* error decreases. As j increases beyond $j = 13$ things get worse. This is (presumably) from rounding error. Note the error represents total failure when $h = 4^{-27}$: we get $f'(x) \approx 0$ instead of $f'(x) = \sinh(1) = 1.1752$. Note $4^{-27} = 5 \times 10^{-17}$ is smaller than $\epsilon_{Matlab} = 2 \times 10^{-16}$.