Selected Assignment # 4 Solutions.

[I graded 2.3 #1, 2.3 #29, 4.1 #1, 4.1 #4, 4.1 #16, problems A, C, F.
Each was worth 5 point for a total of 40.]

2.3 #1. Here I guess all one should say is

$$f(x) = \sqrt{x+4} - 2 = \frac{(x+4) - 2^2}{\sqrt{x+4}+2} = \frac{x}{\sqrt{x+4}+2}.$$

2.3 #29. Using 8 digits the use of the usual form of the quadratic formula produces the two roots $x_+ \approx 10^5$ and $x_- \approx 0$ versus the correct values $x_+ = 99999.99999$ and $x_- = 0.000010000000001$. Thus the approximation has no digits of accuracy for x_-, though the absolute error is modest.

With any precision arithmetic, the best approximations when $a > 0$, $b < 0$, and $c > 0$ (as in this case) are

$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_- = \frac{2c}{-b + \sqrt{b^2 - 4ac}}.$$

4.1 #1. We write down the Lagrange polynomials

$$l_0(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)},$$

$$l_1(x) = \frac{(x)(x-3)(x-4)}{(2)(-1)(-2)},$$

$$l_2(x) = \frac{(x)(x-2)(x-4)}{(3)(1)(-1)},$$

$$l_3(x) = \frac{(x)(x-2)(x-3)}{(4)(2)(1)}.$$

Thus the polynomial which does the job is

$$p(x) = 7\,l_0(x) + 11\,l_1(x) + 28\,l_2(x) + 63\,l_3(x) = \cdots = 7 - 2x + x^3.$$

4.1 #4. Verification is easy.

Now, the theorem on page 139 says that "...there is a unique polynomial p of degree $\le n$ such that $p(x_i) = y_i$ for $0 \le i \le n$". Thus in the case of the given data, there must be one degree *three* polynomial which does the job. There is no problem if there is also a degree *four* polynomial.

4.1 #16. [*I think I would rewrite this question:* "Let $\alpha \in [0,1]$. Consider the polynomial $p(x)$ of degree 2 or less such that $p(0) = 0$, $p(1) = 1$, and $p'(\alpha) = 2$. Show that such a polynomial does not exist for one value α_0 of α. Find the polynomial if $\alpha \ne \alpha_0$."]

Let $p(x) = a_0 + a_1 x + a_2 x^2$. Since $p(0) = 0$, $a_0 = 0$. The other conditions $p(1) = 1$ and $p'(\alpha) = 2$ are two equations for the coefficients:

$$a_1 + a_2 = 1,$$
$$a_1 + 2\alpha a_2 = 2.$$

Since

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 2\alpha \end{pmatrix} = 2\alpha - 1,$$

the pair of equations can not be solved for the equations if $\alpha = 1/2$, which tells us "α_0".

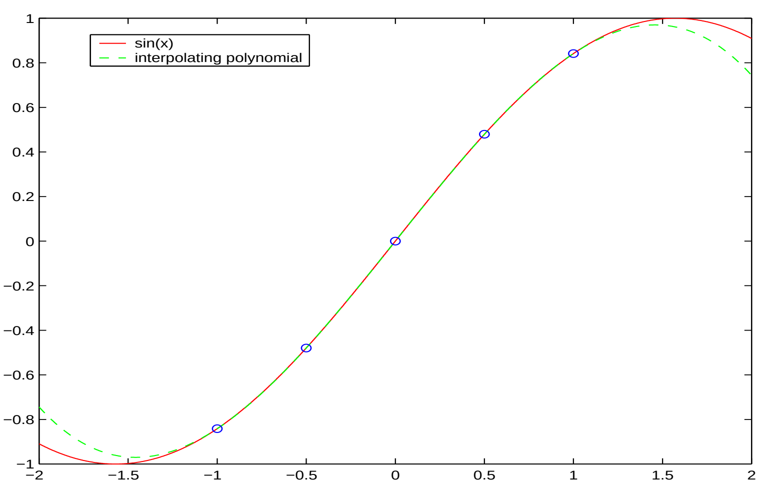
For other values of α,

$$a_0 = 0,\ a_1 = (2\alpha - 2)/(2\alpha - 1),\ \text{and}\ a_2 = 1/(2\alpha - 1).$$

Problem A. My code is

```
n=4; f=inline('sin(x)','x'); a=-1; b=1;
x=linspace(a,b,n+1); y=f(x);
xtest=linspace(a-(b-a)/2,b+(b-a)/2,1000); ytest=f(xtest);
p=polyfit(x,y,n);
pxtest=polyval(p,xtest);
plot(xtest,ytest,'r',xtest,pxtest,'g--',x,y,'o')
legend('sin(x)','interpolating polynomial')
```

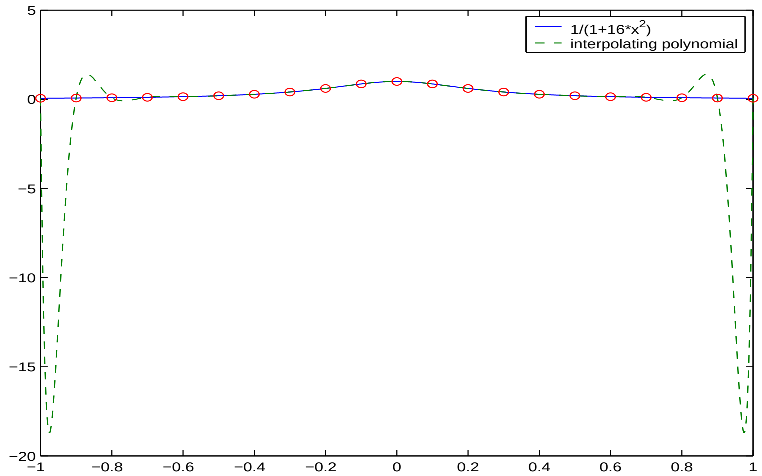
Which gives:



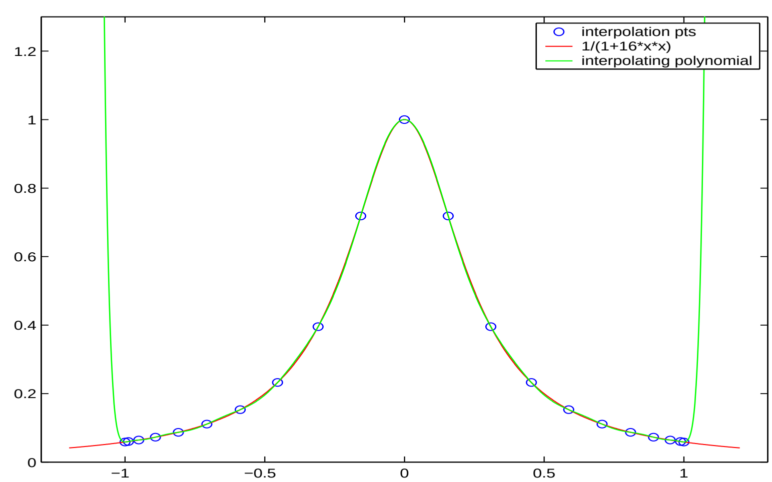
Problem B. [*Not graded.*] A similar code to the above gives:

Problem C. The code and output is:

```
n=20; f=inline('1./(1+16*x.*x)','x'); a=-1; b=1;
len=b-a; j=0:n;
xj=(a+b)/2+(len/2)*cos(j*pi/n); yj=f(xj);
xtest=linspace(a-.2,b+.2,1000); ytest=f(xtest);
```




```
pj=polyfit(xj,yj,n);
pjxtest=polyval(pj,xtest);
plot(xj,yj,'o',xtest,ytest,'r',xtest,pjxtest,'g')
legend('interpolation pts','1/(1+16*x*x)','interpolating polynomial')
axis([-1.3 1.3 0 1.3])
```



Problem D. [*Not graded.*] Note that $n = 20$ in **B** and **C** and f is the same, so the only difference is in the $(x - x_0)\dots(x - x_n)$ part of the error term:

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\dots(x - x_n).$$

That is, for equally spaced x_i versus Chebyshev x_j,

$$\max_{-1\leq x\leq 1}\left|\prod_{j=1}^{20}(x-x_j)\right| \leq \max_{-1\leq x\leq 1}\left|\prod_{i=1}^{20}(x-x_i)\right|,$$

and in fact the difference is great near the ends of the range. See **CP #10** on the next assignment.

Problem F. The first goal is to calculate the Taylor series for $\mathrm{erf}(x)$, so you have some idea what is going on. In fact,

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots$$

by substitution into the series for e^x. Then

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \dots\right),$$

which should explain the given approximation. In fact, use the $n = 4$ case of Taylor's Theorem on $f(x) = \mathrm{erf}(x)$:

$$f(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3}\right) + \frac{f^{(5)}(\xi)}{5!}(x-0)^5$$

because the Taylor series above shows $f^{(4)}(0) = 0$.

Now it requires some work to find $f^{(5)}$, but:

$$f^{(5)}(x) = \frac{8}{\sqrt{\pi}}(3 - 12x^2 + 4x^4)e^{-x^2}.$$

(This is doable by hand or in comparable time by *Mathematica*—a relatively rare instance in which *Mathematica* is effective.)

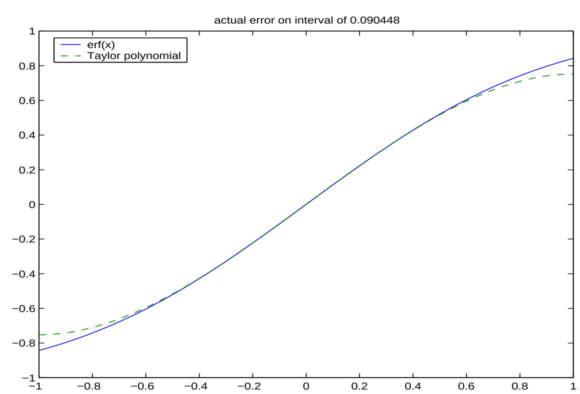
Next, it is easy to see that

$$\left|3 - 12x^2 + 4x^4\right| \leq 5$$

for $x \in [-1, 1]$ (plot or do calculus) and that $e^{-x^2} \leq 1$. Thus

$$|f(x) - [\text{polynomial}]| \leq \frac{8 \cdot 5 \cdot 1}{(\sqrt{\pi})5!} = .188.$$

Compare to the plot (with code following):



```
x=-1:.01:1;
plot(x,erf(x),x,(2/sqrt(pi))*(x-x. 3/3),'--')
legend('erf(x)','Taylor polynomial')
err=max(abs(erf(x)-(2/sqrt(pi))*(x-x. 3/3)));
title(['actual error on interval of ' num2str(err)])
```

Note that the actual error is 0.09, which is roughly half of the predicted error—not bad!