Selected Assignment # 1 Solutions.

3.1 #9. Solve this graphically by seeing where $y = \cos(x)$ and $y = \cos(3x)$ intersect. Answer: $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \dots$

3.1 #15. No. That is, assume as we should that bisection is applied to a continuous f and that $f(a)f(b) < 0$. Then $f(a) \neq 0$. Now there is a smallest solution $r \in (a, b)$. There is at least one number

$$a + k\frac{b-a}{2^n}$$

for some integers k and n, in the interval (a, r), and bisection will choose it or something greater at some stage. Thus at some stage $a_n > a_{n-1}$.

[I can only write such a short and precise answer after drawing pictures and thinking. The main ideas are that on the one hand since $f(a) \neq 0$ and therefore $r > a$, there is a gap between a and r. On the other hand bisection will finds numbers of the type described. I will accept many forms of the correct answer…]

3.1 CP #5. The answer $\lambda = 126.632$ is known so there are more or less realistic methods for finding a bracket $[a, b]$. I started graphing and found that $a = 100, b = 200$ should work. Then I wrote the following *Matlab* program and called it `edbisect.m`:

```
% solves CP #5 in 3.1
% to find a root of
%    f(L) = L cosh(50/L)-L-10

a=100; b=200; % found from plotting
N=17; % adjust appropriately for accuracy

fa=a*cosh(50/a)-a-10;
% I know fb is opposite in sign so no need to calculate it
for j=1:N
   c=(a+b)/2  % no semicolon to see steps
   fc=c*cosh(50/c)-c-10;
   if fc*fa>0
      a=c;
   else
      b=c;
   end
end
[a b] % show last bracket
```

(This is not the perfect program and ideally would use `inline` to define the function, and it should do better checking to see if it is done, etc.)

I ran it and got the bracket $[126.631927490234, 126.632690429688]$ (which compares in correct digits to the answer given) after 17 steps. Note

$$\frac{200 - 100}{2^{18}} \approx 0.0004$$

that is, the third digit after the decimal point is probably correct.

3.2 #1. Here $f(x) = x^2 - R$. Then Newton's is

$$x_{n+1} = x_n - \frac{x_n^2 - R}{2x_n},$$

which simplifies to

$$x_{n+1} = x_n - \frac{x_n}{2} + \frac{R}{2x_n} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

(By the way, this is traditionally called the "mechanics rule for square roots". You can get really good estimates of the square root, from a reasonable guess, by using only one division, one sum, and one halving per step:

```
    >>x=4
x = 4
    >>x=.5*(x+17/x)
x = 4.125
    >>x=.5*(x+17/x)
x = 4.12310606060606
```

which compares to $\sqrt{17} = 4.12310562561766$.)

3.2 #2. To get the given formula, I took the result of **#1**, computed $x_{n+1}^2 - R$ and then found a common denominator—then I saw it:

$$\begin{aligned} x_{n+1}^2 - R &= \frac{1}{4}\left(x_n + \frac{R}{x_n}\right)^2 - R \\ &= \frac{x_n^4 + 2Rx_n^2 + R^2 - 4Rx_n^2}{4x_n^2} \\ &= \left(\frac{x_n^2 - R}{2x_n}\right)^2. \end{aligned}$$

Now to interpret. The definition of quadratic convergence has $e_{n+1} = x_{n+1} - r$ and $e_n = x_n - r$ in it. But here $r = \sqrt{R}$. The difference of squares appears in both sides in the above:

$$(x_{n+1} - \sqrt{R})(x_{n+1} + \sqrt{R}) = \left(\frac{(x_n - \sqrt{R})(x_n + \sqrt{R})}{2x_n}\right)^2.$$

Rearranging I see:

$$x_{n+1} - \sqrt{R} = \left(\frac{x_n + \sqrt{R}}{4x_n^2}\right)(x_n - \sqrt{R})^2,$$

or

$$|e_{n+1}| \le c|e_n|^2$$

if c is the maximum value of $(x_n + \sqrt{R})(4x_n^2)^{-1}$ when x_n is close to $r = \sqrt{R}$. (That is, $c \approx 1/(2\sqrt{R})$.) This is the promised quadratic convergence of Newton's method.

3.2 #9. Now

$$x_{n+1} = x_n - \frac{\sin x_n}{\cos x_n}$$

is Newton's method applied to $f(x) = \sin x$. And $f(x) = 0$ has any integer multiple of π as a solution. Thus with $x_0 = 3$ we certainly hope $x_n \to \pi$. In fact, x_3 has 16 digits of π correct.

(This is actually a very fast converging sequence which computes digits of π. Should you use it to compute the first billion digits of π?)

3.2 #19. If you graph $f(x) = x^2/(1+x^2)$ you see that only $x = 0$ solves $f(x) = 0$ and by thinking about tangent lines, the only way Newton's can converge to zero is to start in some symmetrical neighborhood of zero.

In fact, it had better be the case that $|x_1| < |x_0|$ or else the steps will diverge. Let $x = x_0$ and note that

$$x_1 = x - \frac{x^2}{1+x^2}\frac{(1+x^2)^2}{2x(1+x^2) - x^2(2x)}.$$

Thus we want to find for which x it is true that:

$$\left| x - \frac{x^2}{1+x^2} \frac{(1+x^2)^2}{2x(1+x^2) - x^2(2x)} \right| < |x|.$$

Simplifying we see

$$\left| x - \frac{x^2(1+x^2)}{2x} \right| < |x|,$$

$$\frac{\left|2x^2 - x^2(1+x^2)\right|}{|2x|} < |x|,$$

$$|x|^2|1-x^2| < 2|x|^2,$$

$$-2 < 1 - x^2 < 2,$$

$$-1 < x^2 < 3,$$

or

$$x^2 < 3.$$

That is, $|x_0| < \sqrt{3}$ as given in the back of the book.

3.2 CP #4. In *Matlab* I defined the functions

```
>>f=inline('2*(1-x.^2+x).*(x.*log(x))-x.^2+1','x')
>>df=inline('2*(1-2*x).*x.*log(x)+2*(1-x.^2+x).*(log(x)+1)-2*x','x')
```

Then I plotted on the interval

```
>>x=.01:.01:1; plot(x,f(x)), grid on
```

It is clear there is a solution near .3. And Newton's method gets it quick

```
    >>x=.3
x = 0.30000000000000
    >>x=x-f(x)/df(x)
x = 0.32597756703797
    >>x=x-f(x)/df(x)
x = 0.32795662965170
    >>x=x-f(x)/df(x)
x = 0.32796778497841
    >>x=x-f(x)/df(x)
x = 0.32796778533182
    >>x=x-f(x)/df(x)
x = 0.32796778533182
```

where $x_0 = .3$ has 1 digit, then 2, then 4, then 8, and then all 14 displayed digits.

3.2 CP #14. I wrote the following *Matlab* program and called it `ednewton.m`:

```
% solves CP #14 in 3.2
% to find a root of
%    f(x) = exp(-x.^2)-cos(x)-1
% user sets x=x_0

f=inline('exp(-x.^2)-cos(x)-1','x');
df=inline('-2*x.*exp(-x.^2)+sin(x)','x');
N=10;
for j=1:N
   x=x-f(x)/df(x)
end;
```

Then I ran it with $x_0 = 0$ to see what happens. The slope $f'(0)$ is zero, so I get nonsense ("NAN" in particular). If $x_0 = 1$ then the x_n head out of the interval $[0, 4]$ (and eventually stabilize and slowly converge to $x = 9.42477795246713$ which is very close to $3\pi = 9.42477796076938$).

I plotted f on the interval $[0, 4]$. I saw that there are two solutions to $f(x) = 0$ very close together near $x = 3$. In particular with $x_0 = 3.13$ Newton's gives $x = 3.13108083857007$ after five steps. With $x_0 = 3.15$, Newton's gives $x = 3.15145284329665$ after four steps.