Selected Solutions to Assignment #9

These problems were graded at 3 points each for a total of 27 points.

7.2 #4. An integration-by-parts:

\[
\mathcal{L}\{te^{3t}\}(s) = \int_0^\infty e^{-st}te^{3t}dt = \int_0^\infty te^{(3-s)t}dt = t(3-s)^{-1}e^{(3-s)t}\bigg|_{t=0}^{t=\infty} - \int_0^\infty (3-s)^{-1}e^{(3-s)t}dt
\]

\[
= -(3-s)^{-1}\int_0^\infty e^{(3-s)t}dt = -(3-s)^{-2}e^{(3-s)t}\bigg|_{t=0}^{t=\infty} = +(3-s)^{-2} = \frac{1}{(s-3)^2}.
\]

This function of \(s\) is defined for \(s > 3\); we used “\(s > 3\)” in evaluating the limit at \(t = \infty\). The result agrees with a rule in table 7.1.

7.2 #14. From the table,

\[
\mathcal{L}\{5 - e^{2t} + 6t^2\}(s) = \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}.
\]

7.2 #22. The function is piecewise continuous on \([0, 10]\). The graph is in Figure 1.

7.3 #10. Here we use the last rule in table 7.2 to turn the problem into one covered by the last entry in table 7.1, which means we use the quotient rule:

\[
\mathcal{L}\{te^{2t}\cos 5t\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{e^{2t}\cos 5t\}(s)\right) = -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+5^2}\right) = \frac{(s-2)^2-5^2}{((s-2)^2+5^2)^2}.
\]

7.3 #24. (a) The translation property in question says \(\mathcal{L}\{f\}(s-a) = \mathcal{L}\{e^{at}f\}(s)\). So we write, also using the third line of table 7.1,

\[
\mathcal{L}\{e^{at}t^n\}(s) = \mathcal{L}\{t^n\}(s-a) = \frac{n!}{\sigma^{n+1}}\bigg|_{\sigma=s-a} = \frac{n!}{(s-a)^{n+1}}.
\]

My point in writing “\(\sigma\)” is merely to indicate that we first transform from \(t\) to a new variable and only then replace the transform variable with “\(s-a\)”. 

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**Figure 1.** Sketch of \(f(t)\) from #22 in section 7.2.
The inverse Laplace transform is
\[ \mathcal{L}^{-1}\left\{ e^{at} \right\} (s) = (-1)^n \frac{d^n}{ds^n} \left[ \mathcal{L}^{-1}\left\{ e^{at} \right\} (s) \right] = (-1)^n \frac{d^n}{ds^n} \left( \frac{1}{s-a} \right) \]
\[ = (-1)^n (-1)(-2) \cdots \cdots (-n-1)(-n) \frac{1}{(s-a)^{n+1}} = (-1)^n (-1)^n \frac{n!}{(s-a)^{n+1}}. \]

7.3 #30. Let \( Y(s) = \mathcal{L}\{y\}(s) \) and \( G(s) = \mathcal{L}\{G\}(s) \). To find the transfer function \( H(x) = Y(s)/G(s) \) we assume initial conditions \( y(0) = y'(0) = 0 \) and apply the rules for Laplace transforms of derivatives to the differential equation:
\[ s^2 Y(s) - 0 - 0 + 5(sY(s) - 0) + 6Y(s) = G(s), \]
or \( (s^2 + 5s + 6)Y(s) = G(s) \). Therefore
\[ H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2 + 5s + 6}. \]

7.4 #2. From table 7.1 with \( b = 2 \),
\[ \mathcal{L}^{-1}\left\{ \frac{2}{s^2 + 4} \right\} = \mathcal{L}^{-1}\left\{ \frac{2}{s^2 + 2^2} \right\} = \sin 2t. \]

7.4 #14. Factor the denominator completely. The problem is to find \( A, B, C \) in this equation:
\[ \frac{-8s^2 - 5s + 9}{(s - 2)(s - 1)(s + 1)} = \frac{A}{s - 2} + \frac{B}{s - 1} + \frac{C}{s + 1}. \]
Clear denominators. Substituting \( s = 2 \) gives \( A = -11 \). Substituting \( s = 1 \) gives \( B = 2 \). Substituting \( s = -1 \) gives \( C = 1 \). Thus:
\[ \frac{-8s^2 - 5s + 9}{(s + 1)(s^2 - 3s + 2)} = -\frac{11}{s - 2} + \frac{2}{s - 1} + \frac{1}{s + 1}. \]

7.4 #24: A good example, but not assigned. Note \( s^2 - 4s + 13 \) does not factor over real numbers, but we can complete the square: \( s^2 - 4s + 13 = (s - 2)^2 + 3^2 \). The easiest form is similar to example 7,
\[ \frac{7s^2 - 41s + 84}{((s - 2)^2 + 3^2)(s - 1)} = \frac{A(s - 2) + 3B}{(s - 2)^2 + 3^2} + \frac{C}{s - 1}. \]
Clear denominators. Substitute \( s = 1 \) gives \( C = 5 \). Substituting \( s = 2 \) gives \( 30 = 3B + 9C \) so \( B = -5 \). Substituting \( s = 0 \), for example, gives \( 84 = 2A - 3B + 13C \) so \( A = 2 \). Thus
\[ \frac{7s^2 - 41s + 84}{((s - 2)^2 + 3^2)(s - 1)} = 2 \frac{(s - 2)}{(s - 2)^2 + 3^2} - 5 \frac{3}{(s - 2)^2 + 3^2} + 5 \frac{1}{s - 1}. \]
The inverse Laplace transform is
\[ f(t) = 2e^{2t} \cos 3t - 5e^{2t} \sin 3t + 5e^t. \]

7.4 #26. The partial fraction form is
\[ \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s - 2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2}. \]
Clearing denominators,
\[ 7s^3 - 2s^2 - 3s + 6 = A(s - 2) + Bs(s - 2) + Cs^2(s - 2) + Ds^3. \]
Substitute \( s = 0 \) to find \( A = -3 \). Substitute \( s = 2 \) to find \( D = 6 \). Substitute \( s = 1 \) and \( s = -1 \), for example, to get this system of equations
\[ B + C = 1 \]
\[ B - C = -1 \]
Get \( B = 0 \) and \( C = 1 \). Thus
\[ \mathcal{L}^{-1}\left\{ \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s - 2)} \right\} = \mathcal{L}^{-1}\left\{ -\frac{3}{2} \cdot \frac{2}{s^3} + 0 + \frac{1}{s} + \frac{1}{s - 2} \right\} = 1 - \frac{3}{2}t^2 + 6e^t. \]