Selected Solutions to Assignment #6

These problems were graded at 3 points each for a total of 21 points.
(The Group Project on A#6 is treated as a separate 10 point assignment.)

4.4 #10. The auxiliary equation \( r^2 + 2r - 1 = 0 \) has roots \( r = (-2 \pm \sqrt{4 + 4})/2 = -1 \pm \sqrt{2} \). On the other hand, the right hand side (nonhomogeneity) is \( 10 = 10 t^0 e^{0t} \), so \( s = 0 \) in form (14) because \( r = 0 \) is not a root of the auxiliary equation. (Note you can just check it is not: \( 0^2 + 2(0) - 1 \neq 0 \).) Thus we use

\[
y_p(t) = t^0 (A_0) e^{0t} = A_0.
\]

Substituting into the differential equation gives

\[
(A_0)'' + 2(A_0)' - A_0 = 10
\]

which is 

\[ -A_0 = 10. \]

Thus \( y_p(t) = -10 \).

4.4 #12. The equation is first order but the same undermined coefficients approach works. (You can also find a particular solution by treating this equation as first order linear and using the techniques of section 2.3.) The auxiliary equation is \( 2r + 1 = 0 \). The right hand side is of the form \( 3t^2 = C t^m e^{rt} \) with \( m = 2 \) and \( r = 0 \). Since \( r = 0 \) is not a root (solution) of \( 2r + 1 = 0 \) we have \( s = 0 \). So we try

\[
x_p(t) = t^0 (A_2 t^2 + A_1 t + A_0) e^{0t} = A_2 t^2 + A_1 t + A_0.
\]

Substituting this into \( 2x' + x = 3t^2 \) gives

\[
2(2A_2 t + A_1) + (A_2 t^2 + A_1 t + A_0) = 3t^2.
\]

Matching coefficients of powers gives these three equations:

\[
A_2 = 3,
\]

\[
4A_2 + A_1 = 0,
\]

\[
2A_1 + A_0 = 0.
\]

These are easy to solve, in the given order for instance, to give \( A_0 = 24 \), \( A_1 = -12 \), \( A_2 = 3 \). In fact it is easy to check that \( x_p(t) = 3t^2 - 12t + 24 \) is a solution of \( 2x' + x = 3t^2 \).

4.4 #16. The right side has form \( C t^m e^{at} \sin(\beta t) = t \sin t \) so \( m = 0 \), \( a = 0 \), \( \beta = 1 \). The issue is whether \( r = \alpha \pm i \beta = \pm i \) are roots of the auxiliary equation, which is \( r^2 - 1 = 0 \). But \( r = \pm i \) does not solve \( r^2 - 1 = 0 \). So \( s = 0 \) and we try this form

\[
\theta_p(t) = t^0 (A_1 t + A_0) e^{0t} \cos(1t) + t^0 (B_1 t + B_0) e^{0t} \sin(1t) = (A_1 t + A_0) \cos t + (B_1 t + B_0) \sin t.
\]

Substitution into \( \theta'' - \theta = t \sin t \), and simplification, gives

\[
(-2A_1 t - 2A_0 + 2B_1) \cos t + (-2B_1 t - 2A_1 - 2B_0) \sin t = t \sin t.
\]

The coefficients must match:

\[
-2A_1 = 0,
\]

\[
-2A_0 + 2B_1 = 0,
\]

\[
-2B_1 = 1,
\]

\[
-2A_1 - 2B_0 = 0.
\]

As a (checkable!) result,

\[
\theta_p(t) = -\frac{1}{2} \cos t - \frac{1}{2} t \sin t.
\]
4.4 #22. The right side has form $24t^2e^t = C t^m e^{-rt}$ so $m = 2$ and $r = 1$. The auxiliary equation is $r^2 - 2r + 1 = 0$ and $r = 1$ (appearing on the right side) is a root. Indeed $r^2 - 2r + 1 = (r-1)^2$ so $r = 1$ is a repeated root, and thus $s = 2$. So we try this form

$$x_p(t) = t^2(A_2 t^2 + A_1 t + A_0)e^t.$$ 

Substitution of this form into $x'' - 2x' + x = 24t^2e^t$, and a substantial amount of work (!), gives these three easy equations for $A_2, A_1, A_0$, by matching coefficients of like powers; note that the highest powers of $t$ have coefficient zero: $2A_0 = 0, 6A_1 = 0, 12A_2 = 24$. Thus

$$x_p(t) = t^2(2t^2 + 0t + 0)e^t = 2t^4 e^t.$$ 

This is checkable, worth checking, and checks out!

4.4 #28. (Note only the form of $y_p(t)$ is asked for.) The right side (nonhomogeneity) has form $t^4 e^t$ for $m = 4$ and $r = 1$. The auxiliary equation is $r^2 + 3r - 7 = 0$. Note $1^2 + 3(1) - 7 \neq 0$ so $s = 0$. Thus we should try this form:

$$y_p(t) = (A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0) e^t.$$ 

4.7 #10. Substituting $t^r$ gives characteristic equation

$$r(r - 1) + 2r - 6 = r^2 - r - 6 = 0.$$ 

That is, $(r-3)(r+2) = 0$, so the general solution is

$$y(t) = c_1 t^3 + c_2 t^{-2}.$$ 

At least that was easy . . .

4.7 #46. Here $y_1(t) = t^{-2}$ is given. The other thing we need for reduction of order is $p(t)$. But the equation must be in standard form to know $p(t)$:

$$y'' + \frac{6}{t} y' + \frac{6}{t^2} y = 0.$$ 

Thus $p(t) = 6/t$. Reduction of order is, therefore, this nested pair of integrals:

$$y_2(t) = y_1 \int \frac{e^{-\int \frac{p}{y_1} dt}}{y_1^2} dt = t^{-2} \int \frac{e^{-\int \frac{6}{t} dt}}{t^{-4}} dt = t^{-2} \int t^4 e^{-6 \ln t} dt$$

$$= t^{-2} \int t^4 t^{-6} dt = t^{-2} \int t^{-2} dt = -t^{-3}.$$ 

I have done these integrals quickly, ignoring constants, because we are only looking for one new solution $y_2$. The general solution $c_1 y_1 + c_2 y_2$ will have unknown constants anyway.

In this case we can check the answer two ways. First we may substitute $y_2 = -t^{-3}$ directly to see it is a solution. Second we can notice the ODE is actually a Cauchy-Euler equation.