Selected Solutions to Assignment #10

These problems were graded at 5 points each for a total of 25 points.

7.5 #4. The Laplace transform of the ODE, including the initial values, is
\[
\begin{align*}
[s^2Y(s) - sy(0) - y'(0)] + 6 [sY(s) - y(0)] + 5Y(s) &= \frac{12}{s-1}, \\
(s^2 + 6s + 5)Y(s) &= \frac{12}{s-1} - s + 1, \\
Y(s) &= \frac{12}{s-1} - s + 1 \\
&= \frac{-s^2 + 2s + 11}{s^2 + 6s + 5} = \frac{-s^2 + 2s + 11}{(s-1)(s+1)(s+5)}.
\end{align*}
\]

Partial fractions gives
\[
Y(s) = \frac{-s^2 + 2s + 11}{(s-1)(s+1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}.
\]

Substitution of \(s = 1, s = -1, s = -5\) gives \(A = 1, B = -1, C = -1\), respectively. Thus
\[
Y(s) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}.
\]

It actually is easy to check that this formula gives the solution \(y(t)\) of the initial value problem.

7.5 #16. (This problem is easy because you are not asked to invert the Laplace transform.) Applying the Laplace transform to both sides of the equation,
\[
\begin{align*}
&\quad s^2Y(s) - sy(0) - y'(0) + 6Y(s) = \frac{2}{s^3} - \frac{1}{s} \\
\iff &\quad (s^2 + 6)Y(s) = \frac{2}{s^3} - \frac{1}{s} - 1 \\
&\qquad Y(s) = \frac{2}{s^3} - \frac{1}{s} - 1 \\
&\qquad = \frac{2 - s^2 - s^3}{s^3(s^2 + 6)}.
\end{align*}
\]

7.5 #26. (This one was carefully designed to make the numbers simple.) Apply the Laplace transform and simplify:
\[
\begin{align*}
&\quad [s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 4 [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 6Y(s) = -\frac{12}{s} \\
&\quad (s^3 + 4s^2 + s - 6)Y(s) = s^2 + 8s + 15 - \frac{12}{s} \\
&\quad Y(s) = \frac{s^3 + 8s^2 + 15s - 12}{s(s^3 + 4s^2 + s - 6)} = \frac{s^3 + 8s^2 + 15s - 12}{(s+3)(s+2)(s-1)}.
\end{align*}
\]

At the last stage we have had to factor a cubic: \(s^3 + 4s^2 + s - 6 = (s+3)(s+2)(s-1)\). This is inconvenient.\(^1\) It is efficient to factor by machine. Here is the MATLAB:

\(^1\)It is also unavoidable because the same polynomial must be factored by the previous methods, which gave auxiliary equation \("r^3 + 4r^2 + r - 6 = 0"\).
>> help roots  % so that I can recall how "roots" works
>> roots([1 4 1 -6])
ans =
   -3.0000
   -2.0000
   1.0000

Proceeding, we do partial fractions:

\[ Y(s) = \frac{s^3 + 8s^2 + 15s - 12}{(s + 3)(s + 2)(s - 1)} = \frac{A}{s + 3} + \frac{B}{s + 2} + \frac{C}{s} + \frac{D}{s - 1}. \]

Clearing denominators and substituting \( s = -3, -2, 0, 1 \) gives \( A = 1, B = -3, C = 2, D = 1 \), respectively. The inverse transform is easy, giving

\[ y(t) = e^{-3t} - 3e^{-2t} + 2 + e^t. \]

The initial conditions and the ODE itself are all easy to check. This is the right answer!

**8.1 #2.** We will use the values \( y(0), y'(0), y''(0), \ldots \) to determine the coefficients of the Taylor polynomial at \( x_0 = 0 \). We differentiate the ODE and plug in \( x = 0 \). We stop once we have three nonzero coefficients including \( y(0) = 2 \):

\[
\begin{align*}
y' &= y^2 & \implies & y'(0) = y(0)^2 = 4 \\
y'' &= 2yy' & \implies & y''(0) = 2y'(0)y(0) = 16.
\end{align*}
\]

Thus the quadratic Taylor polynomial approximation which has the correct first three terms is

\[ p_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 = 2 + 4x + 8x^2. \]

**Optional.** We can also solve exactly because the ODE is first-order and separable. The exact solution is \( y(x) = 2/(1 - 2x) \). The Taylor series for the exact solution is

\[ y(x) = \frac{2}{1 - 2x} = 2 + 4x + 8x^2 + \ldots \]

So we got the right first three coefficients.

**8.1 #6.** The method is essentially the same as for #2, but using both initial values \( y(0) = 0, y'(0) = 1 \) to get started:

\[
\begin{align*}
y'' &= -y & \implies & y''(0) = -y(0) = 0 \\
y''' &= -y' & \implies & y'''(0) = -y'(0) = -1 \\
y^{(4)} &= -y'' & \implies & y^{(4)}(0) = -y''(0) = 0 \\
y^{(5)} &= -y''' & \implies & y^{(5)}(0) = -y'''(0) = +1.
\end{align*}
\]

Again we have stopped once we have three nonzero coefficients. The quintic Taylor polynomial which has the correct first three nonzero coefficients is

\[ p_5(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5. \]

**Optional.** We can also solve exactly because the ODE is constant-coefficient and linear and homogeneous. In fact it is very quick to see that the exact solution is

\[ y(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots \]