Exam # 2 SOLUTIONS.

1 (a). \( \mathbf{v}(t) = 2t \mathbf{i} + \cos(\pi t) \mathbf{j} + \mathbf{k}, \)
\( a(t) = 2 \mathbf{i} - \pi \sin(\pi t) \mathbf{j}. \)

1 (b). At \( t = 1, \mathbf{v}(1) = 2 \mathbf{i} - \mathbf{j} + \mathbf{k}, a(1) = 2 \mathbf{i}. \) Thus,
\[ \theta = \cos^{-1} \left( \frac{\mathbf{v}(1) \cdot a(1)}{|\mathbf{v}(1)||a(1)|} \right) = \cos^{-1} \left( \frac{4}{\sqrt{6} \cdot 2} \right) = \cos^{-1} \left( \frac{2}{\sqrt{6}} \right). \]

2. (a) \( \leftrightarrow \) II; (b) \( \leftrightarrow \) I; (c) \( \leftrightarrow \) III; (d) \( \leftrightarrow \) IV.

3 (a). First, \( \mathbf{v} = -\sin t \mathbf{i} + \cos t \mathbf{j} + 2 \mathbf{k} \) so \( |\mathbf{v}| = \sqrt{5}. \) Thus \( s(t) = \int_0^t |\mathbf{v}(\tau)| \, d\tau = \int_0^t \sqrt{5} \, d\tau = \sqrt{5} t. \)

3 (b). We need \( \mathbf{T} \) and \( |d\mathbf{T}/dt|. \) Since \( |\mathbf{v}| = \sqrt{5} \) it follows that \( \mathbf{T} = \frac{1}{\sqrt{5}} (-\sin t \mathbf{i} + \cos t \mathbf{j} + 2 \mathbf{k}). \) Thus
\[ \frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{5}} (-\cos t \mathbf{i} - \sin t \mathbf{j}); \]
\[ \kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{5}} = \frac{1}{5}. \]
5. By the fundamental theorem of calculus I,
\[ \frac{\partial f}{\partial y} = e^{-y^2}; \quad \frac{\partial f}{\partial x} = -e^{-x^2}. \]

6. We need the direction of \( \mathbf{v} \) and also the gradient of \( f \). Then we can take the directional derivative:
\[ \hat{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \begin{pmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{pmatrix}; \quad \nabla f = i + 3y^2j + 2z k; \quad \nabla f|_{(0,1,2)} = i + 3j + 4k; \quad D_{\hat{u}}f = \nabla f|_{(0,1,2)} \cdot \hat{u} = -\frac{13}{5}. \]

[Note that the answer is a scalar not a vector!]

7. Here \( f(x, y, z) = x^2 + y^2 + z^2 \) and the surface is a level surface of \( f \). The gradient will be the normal vector to the plane we want:
\[ \nabla f = 2xi + 2yj + 2zk; \quad \mathbf{n} = \nabla f|_{(1,1,-1)} = 2i + 2j - 2k. \]

The plane is \( \mathbf{n} \cdot \mathbf{PP}_0 = 0: \)
\[ 2(x - 1) + 2(y - 1) - 2(z + 1) = 0 \]

(or \( x + y - z = 3 \)).

8. The chain rule is
\[ \frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} \]
and the diagram is at right:

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**Extra Credit.** In this example we can do without calculus. In fact, the curve \( \mathbf{r}(t) \) can be drawn onto the level curves of \( f \) and you will see the answer. Note that the curve is \( x = t, y = t^2 + 1 \) or equivalently \( y = x^2 + 1 \). Then we see from the level curves of \( f \) that the minimum is at the point \( (x = 0, y = 1) \), which happens when \( t = 1 \), and the value of \( f \) is 1. See the picture for 4 above.

Of course, calculus can also be used!