Solutions to Quiz #8

1. By the Ratio test,
\[
\lim_{n \to \infty} \frac{|x-2|^{n+1}}{(n+1)^2} = \lim_{n \to \infty} \frac{|x-2| \cdot n^2}{(n+1)^2} = |x-2| \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = |x-2| < 1
\]
implies (absolute) convergence. That is, the interval of convergence is
\[
|x-2| < 1 \iff -1 < x - 2 < 1 \iff 1 < x < 3,
\]
ignoring endpoints. Thus the radius of convergence is 1.

2. We now consider the two endpts:
\[
x = 1: \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges absolutely (see } x = 3 \text{ below), and}
\]
\[
x = 3: \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges absolutely because it is a p-series with } p = 2.
\]

3. By the Ratio test,
\[
\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{|x| \cdot n!}{(n+1)! \cdot |x|^n} = |x| \lim_{n \to \infty} \frac{1}{n+1} = 0,
\]
which is less than one independent of the value of \(x\). Thus convergence happens for all \(x\), that is, the interval of convergence is \((-\infty, \infty) = \mathbb{R}\).

4. (I was expecting everyone would do 4 by differentiation and 5 by integration, but even if you did not, I graded appropriately.)

By differentiating,
\[
-\sin x = \frac{d}{dx} \cos x = 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots,
\]
so
\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

5. Here
\[
\int \cos x \, dx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + C,
\]
that is, you do get the series for \(\sin x\) but with an additional constant of integration. But of course \(\int \cos x \, dx\) and \(\sin x \, dx\) do differ by an unknown constant of integration.