Solutions to Quiz #7

1. $\sum_{n=1}^\infty \ln(1/n)$ *diverges.* In fact,

$$\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$$

because $\lim_{n\to\infty} 1/n = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, so the terms a_n do not have limit zero. That is, $\lim a_n \neq 0$ and the nth term test show the series $\sum a_n$ diverges.

2. $\sum_{n=2}^\infty 7/4^n$ converges, and the sum is

$$\sum_{n=2}^\infty \frac{7}{4^n} = \frac{7/(4^2)}{1-(1/4)} = \frac{7}{12}.$$

(Note the series is geometric, $a = 7/(4^2)$, and $r = 1/4 < 1$.)

3. Consider the partial sum

$$S_N = \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{(N+1)(N+2)}.$$

We do partial fractions on the general term $\frac{1}{(n+1)(n+2)}$:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \qquad \iff \qquad 1 = (A+B)n + (2A+B),$$

which shows $A = 1$ and $B = -1$. Now

$$S_N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{N+1} - \frac{1}{N+2} = \frac{1}{2} - \frac{1}{N+2},$$

by lots of cancellation. Thus

$$\sum_{n=1}^\infty \frac{1}{(n+1)(n+2)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2} - \frac{1}{N+2} = \frac{1}{2} - 0 = \frac{1}{2}.$$

4. The series converges by the integral test. In fact,

$$\sum_{n=1}^\infty \frac{e^n}{1+e^{2n}} \text{ converges} \iff \int_1^\infty \frac{e^x\,dx}{1+e^{2x}} \text{ converges}$$

by the integral test. And the substitution $u = e^x$ gives:

$$\int_1^\infty \frac{e^x\,dx}{1+e^{2x}} = \lim_{b\to\infty} \int_1^b \frac{e^x\,dx}{1+(e^x)^2} = \lim_{b\to\infty} \int_e^{e^b} \frac{du}{1+u^2}$$

$$= \lim_{b\to\infty} \tan^{-1}(u)\Big|_e^{e^b} = \frac{\pi}{2} - \tan^{-1}(e),$$

that is, the integral converges.