Solutions to Quiz #7

1. $\sum_{n=1}^{\infty} \ln \left( \frac{1}{n} \right)$ diverges. In fact,

$$\lim_{n \to \infty} \ln \left( \frac{1}{n} \right) = -\infty$$

because $\lim_{n \to \infty} 1/n = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$, so the terms $a_n$ do not have limit zero. That is, $\lim a_n \neq 0$ and the $n$th term test show the series $\sum a_n$ diverges.

2. $\sum_{n=2}^{\infty} \frac{7}{4^n}$ converges, and the sum is

$$\sum_{n=2}^{\infty} \frac{7}{4^n} = \frac{7/(4^2)}{1 - (1/4)} = \frac{7}{12}.$$  

(Note the series is geometric, $a = 7/(4^2)$, and $r = 1/4 < 1$.)

3. Consider the partial sum

$$S_N = \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(N+1)(N+2)}.$$  

We do partial fractions on the general term $\frac{1}{(n+1)(n+2)}$:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \iff 1 = (A + B)n + (2A + B),$$

which shows $A = 1$ and $B = -1$. Now

$$S_N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N+1} - \frac{1}{N+2} = \frac{1}{2} - \frac{1}{N+2},$$

by lots of cancellation. Thus

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{1}{2} - \frac{1}{N+2} = \frac{1}{2} - 0 = \frac{1}{2}.$$  

4. The series converges by the integral test. In fact,

$$\sum_{n=1}^{\infty} \frac{e^n}{1 + e^{2n}} \text{ converges } \iff \int_{1}^{\infty} \frac{e^x}{1 + e^{2x}} \text{ converges},$$

by the integral test. And the substitution $u = e^x$ gives:

$$\int_{1}^{\infty} \frac{e^x}{1 + e^{2x}} \, dx = \lim_{b \to \infty} \int_{1}^{e^b} \frac{e^x}{1 + (e^x)^2} \, dx = \lim_{b \to \infty} \int_{e}^{b} \frac{du}{1 + u^2} = \lim_{b \to \infty} \tan^{-1}(u) \bigg|_{e}^{b} = \frac{\pi}{2} - \tan^{-1}(e),$$

that is, the integral converges.