Quiz # 5 Solutions

1. (I screwed up this problem with a typo! It *should* have been "$\int_2^\infty \frac{dx}{x(\ln x)^2}$," which is only improper at infinity and much easier. Therefore NOT GRADED.)

Solution to the problem "as is": The integrand is positive. Because of the vertical asymptotes of the integrand $y = \frac{dx}{x(\ln x)^2}$ at $x = 0$ and at $x = 1$, the integral is improper at $x = 0$ *and* at $x = 1$ *and* as $x \to \infty$. We will see it diverges because of that, as follows:

$$\int_0^\infty \frac{dx}{x(\ln x)^2} \geq \int_1^2 \frac{dx}{x(\ln x)^2} = \lim_{a\to 1^+} \int_a^2 \frac{dx}{x(\ln x)^2} = \lim_{a\to 1^+} \int_{\ln a}^{\ln 2} \frac{du}{u^2} = \lim_{a\to 1^+} \frac{1}{\ln a} - \frac{1}{\ln 2} = +\infty$$

where I have used the substitution $u = \ln x$, of course.

Solution to the problem as it should have been: Substitute $u = \ln x$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} \frac{-1}{u}\Big]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln b} = \frac{1}{\ln 2}.$$

2. By L'Hopital's rule, applied twice:

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x/2)} \overset{\text{L'H ``}\frac{0}{0}\text{''}}{=} \lim_{x\to 0} \frac{2x}{(1/2)\sin(x/2)} = \lim_{x\to 0} \frac{4x}{\sin(x/2)} \overset{\text{L'H ``}\frac{0}{0}\text{''}}{=} \lim_{x\to 0} \frac{8}{\cos(x/2)} = 8.$$

3. Rewrite as a fraction and apply L'Hopital:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\text{L'H ``}\frac{\infty}{\infty}\text{''}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = 0.$$

4. (a) The integral $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$ is improper because $\lim_{x\to 0} \frac{1}{\sqrt{|x|}} = +\infty$, that is, there is a vertical asymptote of the integrand at $x = 0$.

(b) We split into two relatively easy improper integrals:

$$\begin{aligned}
\int_{-1}^4 \frac{dx}{\sqrt{|x|}} &= \int_{-1}^0 \frac{dx}{\sqrt{|x|}} + \int_0^4 \frac{dx}{\sqrt{|x|}} \\
&= \int_{-1}^0 \frac{dx}{\sqrt{-x}} + \int_0^4 \frac{dx}{\sqrt{x}} \\
&= \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{a\to 0^+} \int_a^4 \frac{dx}{\sqrt{x}} \\
&= \lim_{b\to 0^-} -2(-x)^{1/2}\Big]_{-1}^b + \lim_{a\to 0^+} 2x^{1/2}\Big]_a^4 \\
&= \lim_{b\to 0^-} 2\sqrt{1} - 2\sqrt{-b} + \lim_{a\to 0^+} 2\sqrt{4} - 2\sqrt{a} \\
&= 2 + 4 = 6.
\end{aligned}$$