Solutions to MIDTERM EXAM # 2

1. Use L'Hopital's rule to evaluate the limit $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Solution. I use L'Hopital's rule twice, even though in the second limit it is really unnecessary as you should recognize the limit:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{n\to\infty} \frac{\sin x}{2x} = \lim_{n\to\infty} \frac{\cos x}{2} = \frac{1}{2}.$$

2. Apply the integral test or the direct comparison test to determine if the series converges or diverges: $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Solution. (By Integral Test). Using the substitution $u = \ln x$,

$$\int_2^\infty \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u\,du = \lim_{b\to\infty} \frac{(\ln b)^2 - (\ln 2)^2}{2} = \infty,$$

that is, the integral diverges. Thus the series *diverges.*

Solution. (By Direct Comparison). Note $\ln n > 1$ for $n \ge 3$ so

$$\frac{\ln n}{n} \ge \frac{1}{n}$$

for $n \ge 3$. Furthermore, the harmonic series (p-series with $p = 1$) $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges. Thus the series in question is bigger than a divergent series, and thus *diverges.*

3. (a) Explain why $\int_0^1 \frac{4x\,dx}{1-x^2}$ is improper.

Solution. The integral is improper because the integrand $\frac{4x}{1-x^2}$ is unbounded on the interval $(0, 1)$, and in fact $\lim_{x\to 1^-} \frac{4x}{1-x^2} = +\infty$.

(b) Evaluate the improper integral.

Solution. With the substitution $u = 1 - x^2$,

$$\int_0^1 \frac{4x\,dx}{1-x^2} = \lim_{b\to 1^-} \int_0^b \frac{4x\,dx}{1-x^2} = \lim_{b\to 1^-} \int_1^{1-b^2} \frac{-2\,du}{u} = -2 \lim_{b\to 1^-} \ln(1-b^2) - \ln 1 = -2 \lim_{z\to 0^+} \ln z = +\infty.$$

The last step uses the fact that if $b \to 1^-$ then $1 - b^2 \to 0^+$. Thus the integral diverges (to $+\infty$).

4. Find the limit of the sequence $a_n = n \tan\left(\frac{1}{n}\right)$.

Solution. By L'Hopital at the second equality,

$$\lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\frac{-1}{n^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = 1.$$

The last step uses the idea that if $z \to 0$ then $\sec z \to 1$ because $\sec z = 1/\cos z$.

5. (a) Does the sequence $a_n = \frac{(-1)^n}{4(3^n)}$ converge? If so, find its limit.

Solution. Yes, it converges to zero because $(-1/3)^n \to 0$ as $n \to \infty$.

(b) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{4(3^n)}$ converge? If so, find its sum.

Solution. Yes, it *converges.* In fact it is geometric with $r = \frac{-1}{3}$, so that $|r| < 1$. Also $a = -1/12$—substitute $n = 1$ in $\frac{(-1)^n}{4(3^n)}$. Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4(3^n)} = \frac{-1/12}{1-(-1/3)} = -\frac{1}{16}.$$

6. Does the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converge or diverge? Explain. Include name(s) of test(s) used.

Solution. It *converges* by comparison to $\sum \frac{1}{n^2}$, which is the series one gets if one ignores the "-1". In fact, by the Limit Comparison Test,

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}} = \sqrt{1} = 1,$$

which is neither 0 nor ∞, so the two series do the same thing. But $\sum \frac{1}{n^2}$ is a convergent p-series with $p = 2$.

Solution. It converges by integral test because

$$\int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to\infty} \sec^{-1} x \Big]_2^b = \lim_{b\to\infty} \sec^{-1} b - \sec^{-1} 2 = \frac{\pi}{2} - \frac{\pi}{3}$$

is finite (i.e. converges).

7. Does the series $\sum_{n=2}^{\infty} \frac{3}{(n-1)n}$ converge or diverge? If it converges, find its sum.

NOTE. I asked for the sum! Thus the series must be either geometric or telescoping and it's not geometric.

Solution. The series is telescoping. By partial fractions,

$$\frac{3}{(n-1)n} = \frac{3}{n-1} - \frac{3}{n}.$$

The Nth partial sum is

$$S_N = \sum_{n=2}^{N} \frac{3}{(n-1)n} = \frac{3}{1\cdot 2} + \frac{3}{2\cdot 3} + \cdots + \frac{3}{(N-1)\cdot N} = \frac{3}{1} - \frac{3}{2} + \frac{3}{2} - \frac{3}{3} + \cdots + \frac{3}{N-1} - \frac{3}{N} = 3 - \frac{3}{N},$$

so

$$\sum_{n=2}^{\infty} \frac{3}{(n-1)n} = \lim_{N\to\infty} S_N = 3 - 0 = 3$$

converges. (The series converges because the limit of the partial sums exists, not because of some test; you may apply a comparison test to show convergence but that doesn't get the sum.)

8. Does the series $\sum_{n=0}^{\infty} \frac{n^2\,2^n}{n!}$ converge or diverge? Explain. Include name(s) of test(s) used.

Solution. The series *converges* by the Ratio test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2\,2^{n+1}}{(n+1)!}}{\frac{n^2\,2^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^2\,2^{n+1}\cdot n!}{(n+1)!\cdot n^2\,2^n} = \lim_{n\to\infty} \frac{(n+1)^2\,2}{(n+1)\,n^2} = \lim_{n\to\infty} \frac{2(n+1)}{n^2} = 0 = R,$$

and $|R| < 1$.

Extra Credit. Does the series $\sum_{n=1}^{\infty} \frac{\tan(\frac{1}{n})}{n}$ converge or diverge? Carefully explain.

Solution. The series *converges* by Limit Comparison to the series $\sum \frac{1}{n^2}$. In fact,

$$\lim_{n\to\infty} \frac{\frac{\tan(\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2 \tan\left(\frac{1}{n}\right)}{n} = \lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = 1.$$

The last step is the result of problem **4**. As the limit just computed is neither zero nor infinity, the two series do the same thing. On the other hand, $\sum \frac{1}{n^2}$ is a convergent p-series.