Math 201 Calculus II (Bueler) October 22, 2003

Solutions to Midterm Exam #1

1. Substitute $u = \tan\theta$:

$$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right) \sec^2\theta \, d\theta = \int_0^1 1 + e^u \, du = u + e^u\Big]_0^1 = (1+e) - (0+1) = e.$$

2. Rewrite the differential equation as an integral and recognize that the integrand is the derivative of $\sec^{-1} x$:

$$y = \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + c,$$

at least when x is positive. Now use the data $x = 2$, $y = \pi$:

$$\pi = \sec^{-1} 2 + c \qquad \Longleftrightarrow \qquad c = \pi - \sec^{-1} 2 = \pi = \frac{\pi}{3} = \frac{2\pi}{3}.$$

Thus $y = \sec^{-1}(x) + 2\pi/3$.

3. **(b)** By washers, that is, discs, $dV = \pi r^2 \, dx = \pi \left(\frac{1}{\sqrt{x}}\right)^2 dx$. Therefore

$$V = \int_1^2 \pi \left(\frac{1}{\sqrt{x}}\right)^2 dx = \pi \int_1^2 \frac{1}{x} \, dx = \pi(\ln 2 - \ln 1) = \pi \ln 2.$$

4. First,

$$y = 3^{(\cos^{-1} x)} = e^{(\ln 3)(\cos^{-1} x)}.$$

Thus by the chain rule

$$\frac{dy}{dx} = e^{(\ln 3)(\cos^{-1} x)} \, (\ln 3) \frac{-1}{\sqrt{x^2-1}} = \frac{-(\ln 3)\left(3^{(\cos^{-1} x)}\right)}{\sqrt{x^2-1}}.$$

5. Recall $\sinh x = \frac{1}{2}(e^x - e^{-x})$ while $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Thus

$$\frac{d}{dx}(\sinh x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

6. Do integration-by-parts twice (with $u = t^2$, $dv = e^{3t}\,dt$ in the first case and $u = t$, $dv = e^{3t}\,dt$ in the second):

$$\int t^2 e^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{3} \int t e^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{9} t e^{3t} + \frac{2}{9} \int e^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{9} t e^{3t} + \frac{2}{27} e^{3t} + C.$$

7. Partial fractions is recommended (though completing the square will actually work). Note $x^2 + 5x - 6 = (x + 6)(x - 1)$. Therefore

$$\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1} \qquad \Longleftrightarrow \qquad A = \frac{2}{7} \text{ and } B = \frac{5}{7}.$$

It follows that

$$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{1}{x+6}\,dx + \frac{5}{7}\int \frac{1}{x-1}\,dx = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C.$$

8. The trig. substitution $2x = \tan\theta$ (and $dx = (1/2)\sec^2\theta\,d\theta$) gives

$$\begin{aligned}\int \frac{8\,dx}{(4x^2+1)^2} &= \int \frac{4\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = 4\int \cos^2\theta\,d\theta = 2\int 1+\cos 2\theta\,d\theta \\ &= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = 2\left(\theta + \sin\theta\cos\theta\right) + C \\ &= 2\left(\tan^{-1}(2x) + \frac{2x}{\sqrt{4x^2+1}}\frac{1}{\sqrt{4x^2+1}}\right) + C = 2\tan^{-1}(2x) + \frac{4x}{4x^2+1} + C.\end{aligned}$$

These steps involv both trig. identities and the drawing of a right triangle with angle θ, opposite side $2x$, and adjacent side 1.

Extra Credit. This problem is hard in that a long sequence of standard steps need to be applied. In particular:

> Multiply both top and bottom by e^x; then substitute $u = e^x$; then factor $u^3+1 = (u+1)(u^2-u+1)$ and do partial fractions; then complete the square $u^2 - u + 1 = (u - 1/2)^2 + 3/4$; then substitute $v = u - 1/2$ and split the integral; find antiderivatives; return all the way to x.

Here goes:

$$\begin{aligned}\int \frac{dx}{e^{2x}+e^{-x}} &= \int \frac{e^x dx}{e^{3x}+1} = \int \frac{du}{u^3+1} = \int \frac{-1}{u+1} + \frac{2}{u^2-u+1}\,du \\ &= -\ln|u+1| + \int \frac{(u-1/2)+5/2}{(u-1/2)^2+3/4}\,du \\ &= -\ln(e^x+1) + \int \frac{v}{v^2+3/4}\,dv + \frac{5}{2}\int \frac{1}{v^2+3/4}\,dv \\ &= -\ln(e^x+1) + \frac{1}{2}\ln(v^2+3/4) + \frac{5}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}v\right) + C \\ &= -\ln(e^x+1) + \frac{1}{2}\ln\left((e^x-1/2)^2+3/4\right) + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{2e^x-1}{\sqrt{3}}\right) + C.\end{aligned}$$

(Yes, extra credit points are really harder to get than regular points...)