

Solutions to Midterm Exam #1

1. Substitute \( u = \tan \theta \):
   \[
   \int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta = \int_0^1 1 + e^u \, du = u + e^u \bigg|_0^1 = (1 + e) - (0 + 1) = e.
   \]

2. Rewrite the differential equation as an integral and recognize that the integrand is the derivative of \( \sec^{-1} x \):
   \[
   y = \int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + c, \]
   at least when \( x \) is positive. Now use the data \( x = 2, y = \pi \):
   \[
   \pi = \sec^{-1} 2 + c \iff c = \pi - \sec^{-1} 2 = \pi - \frac{2\pi}{3} = \frac{-\pi}{3}.
   \]
   Thus \( y = \sec^{-1}(x) + 2\pi/3 \).

3. (b) By washers, that is, discs, \( dV = \pi r^2 \, dx = \pi \left( \frac{1}{\sqrt{x}} \right)^2 \, dx \). Therefore
   \[
   V = \int_1^2 \pi \left( \frac{1}{\sqrt{x}} \right)^2 \, dx = \pi \int_1^2 \frac{1}{x} \, dx = \pi (\ln 2 - \ln 1) = \pi \ln 2.
   \]

4. First,
   \[
   y = 3^{(\cos^{-1} x)} = e^{(\ln 3)(\cos^{-1} x)}.
   \]
   Thus by the chain rule
   \[
   \frac{dy}{dx} = e^{(\ln 3)(\cos^{-1} x)} (\ln 3) \frac{-1}{\sqrt{x^2 - 1}} = \frac{-3}{\sqrt{x^2 - 1} \sqrt{x^2}}.
   \]

5. Recall \( \sinh x = \frac{1}{2}(e^x - e^{-x}) \) while \( \cosh x = \frac{1}{2}(e^x + e^{-x}) \). Thus
   \[
   \frac{d}{dx} (\sinh x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.
   \]

6. Do integration-by-parts twice (with \( u = t^2, \, dv = e^{3t} \, dt \) in the first case and \( u = t, \, dv = e^{3t} \, dt \) in the second):
   \[
   \int t^2 e^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{3} \int te^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{9} t e^{3t} + \frac{2}{9} \int e^{3t} \, dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{9} t e^{3t} + \frac{2}{27} e^{3t} + C.
   \]
7. Partial fractions is recommended (though completing the square will actually work). Note \( x^2 + 5x - 6 = (x + 6)(x - 1) \). Therefore

\[
\frac{x + 4}{(x + 6)(x - 1)} = \frac{A}{x + 6} + \frac{B}{x - 1} \iff A = \frac{2}{7} \text{ and } B = \frac{5}{7}
\]

It follows that

\[
\int \frac{x + 4}{x^2 + 5x - 6} \, dx = \frac{2}{7} \int \frac{1}{x + 6} \, dx + \frac{5}{7} \int \frac{1}{x - 1} \, dx = \frac{2}{7} \ln |x + 6| + \frac{5}{7} \ln |x - 1| + C.
\]

8. The trig. substitution \( 2x = \tan \theta \) (and \( dx = (1/2) \sec^2 \theta \, d\theta \)) gives

\[
\int \frac{8 \, dx}{(4x^2 + 1)^2} = \int \frac{4 \sec^2 \theta}{(\sec^2 \theta)^2} \, d\theta = 4 \int \cos^2 \theta \, d\theta = 2 \int 1 + \cos 2\theta \, d\theta = 2 \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = 2 \left( \theta + \sin \theta \cos \theta \right) + C = 2 \tan^{-1}(2x) + \frac{4x}{4x^2 + 1} + C.
\]

These steps involve both trig. identities and the drawing of a right triangle with angle \( \theta \), opposite side \( 2x \), and adjacent side \( 1 \).

**Extra Credit.** This problem is hard in that a long sequence of standard steps need to be applied. In particular:

Multiply both top and bottom by \( e^x \); then substitute \( u = e^x \); then factor \( u^3 + 1 = (u + 1)(u^2 - u + 1) \) and do partial fractions; then complete the square \( u^2 - u + 1 = (u - 1/2)^2 + 3/4 \); then substitute \( v = u - 1/2 \) and split the integral; find antiderivatives; return all the way to \( x \).

Here goes:

\[
\int \frac{dx}{e^{2x} + e^{-x}} = \int \frac{e^x dx}{e^{3x} + 1} = \int \frac{du}{u^3 + 1} = \int -\frac{1}{u + 1} + \frac{2}{u^2 - u + 1} \, du
\]

\[
= -\ln |u + 1| + \int \frac{(u - 1/2) + 5/2}{(u - 1/2)^2 + 3/4} \, du
\]

\[
= -\ln(e^x + 1) + \frac{1}{2} \ln(u^2 + 3/4) + \frac{5}{2} (\frac{1}{\sqrt{3}} \tan^{-1}(\frac{2v}{\sqrt{3}})) + C
\]

\[
= -\ln(e^x + 1) + \frac{1}{2} \ln ((e^x - 1/2)^2 + 3/4) + \frac{5}{\sqrt{3}} \tan^{-1}(\frac{2e^x - 1}{\sqrt{3}}) + C.
\]

(Yes, extra credit points are really harder to get than regular points...)}