Midterm Exam # 1 Solutions

1. *Something like*: “The values of \( f(x) \) can be made to be as close as one wishes to \( L \) by choosing \( x \) sufficiently close to, but not equal to, \( a \).”

2. (a) 
\[
(f \circ g)(x) = \sin \left( \frac{1}{1+x} \right),
\]
\[
(f \circ f)(x) = \sin(\sin x),
\]
\[
(g \circ f)(x) = \frac{1}{1 + \sin x}.
\]

(b) 
- \( \text{domain } f \circ g = \{ x \neq 1 \} \),
- \( \text{domain } f \circ f = \{ \text{all real numbers} \} \),
- \( \text{domain } g \circ f = \{ \sin x \neq 1 \} = \{ x \text{ is not one of } \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots \} \).

3. (a) \( f'(x) = -2x + 6x^5 - 18x^{17} \).

(b) 
\[
g'(x) = \frac{2x(x-1) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.
\]

4. The original limit gives the indeterminant form “\( \infty - \infty \)”. Thus we complete the difference of squares:
\[
\lim_{x \to \infty} \sqrt{x^2 + 1} - x = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{(\sqrt{x^2 + 1} + x)} = \lim_{x \to \infty} \frac{(x^2 + 1) - x^2}{(\sqrt{x^2 + 1} + x)}
\]
\[
= \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x} = 0.
\]

5. [Sketch will be given in class.]

6. 
\[
\lim_{h \to 0} \frac{(1 + h)^4 - 1}{h} = \lim_{h \to 0} \frac{1 + 4h + 6h^2 + 4h^3 + h^4 - 1}{h} = \lim_{h \to 0} \frac{h(4 + 6h + 4h^2 + h^3)}{h}
\]
\[
= \lim_{h \to 0} (4 + 6h + 4h^2 + h^3) = 4 + 0 + 0 + 0 = 4.
\]

7. [Sketch will be given in class.]

8. [Sketch will be given in class.]
9. First,
\[ \frac{dy}{dx} = 3 + (-1)x^{-2} = 3 - \frac{1}{x^2}. \]
[Note that you may use the power rule to calculate the derivative. The definition of the derivative also works, of course, but if I want you to use it I will definitely say “Use the definition of the derivative to . . . ”] Thus
\[ m = \left. \frac{dy}{dx} \right|_{x=1} = 3 - 1 = 2 \]
so the tangent line is
\[ y - 4 = 2(x - 1) \quad \text{or} \quad y = 2x + 2 \]
because we know a point \((x_0, y_0) = (1, 4)\).

Extra Credit. The point here is that as one approaches the origin from either left or right on the graph \(y = x^{2/3}\), the slope goes to infinity. [It is not enough to note that the slope is undefined at \(x = 0\), because there are plenty of graphs like \(y = |x|\) where the slope is undefined at \(x = 0\) but no one would say “the tangent line is vertical.”] The calculations which support my claim that the slope goes to infinity are:
\[ m_{\text{tangent, right}} = \lim_{h \to 0^+} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0^+} h^{-1/3} = +\infty, \]
\[ m_{\text{tangent, left}} = \lim_{h \to 0^-} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0^-} h^{-1/3} = -\infty, \]
so
\[ |m_{\text{tangent, right}}| = |m_{\text{tangent, left}}| = +\infty \]
so the tangent line at the origin is vertical (even though the derivative at \(x = 0\) does not exist).