Quiz # 3 Solutions

1. I rewrite “cos \( x = x \)” as the equivalent equation \( \cos x - x = 0 \). The function \( f(x) = \cos x - x \) is continuous. And

\[
\begin{align*}
  f(0) &= \cos 0 - 0 = 1 \\
  f(1) &= \cos 1 - 1.
\end{align*}
\]

Now, I don’t know what \( \cos 1 \) is, but it is not 1 and it is less than 1. (The only inputs for which \( \cos x \) gives value one are the even multiples of \( \pi \), and 1 is not an even multiple of \( \pi \).) Thus \( \cos 1 - 1 < 0 \).

So

\[
\begin{align*}
  f(0) &= 0 > 0 \\
  f(1) &= 1 < 0
\end{align*}
\]

and \( f \) is continuous on \((0,1)\). The Intermediate Value Theorem shows there is some \( x \) for which \( f(x) = 0 \), that is, there is a solution to the equation \( \cos x = x \) on \((0,1)\).

(It turns out that \( x \approx 0.739085133215 \) or so, but that is irrelevant to the question.)

2. (a) For \( x \neq 1 \) the given function simplifies:

\[
f(x) = \frac{x^2 - x}{x^2 - 1} = \frac{x(x - 1)}{(x - 1)(x + 1)} = \frac{x}{x + 1}.
\]

(I emphasize that this calculation is correct for \( x \) not equal to 1.) Thus

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}.
\]

Unfortunately, \( f(1) = 1 \) as stated in the problem. Thus

\[
\lim_{x \to a} f(x) \neq f(a)
\]

when \( a = 1 \). Thus \( f \) is not continuous at \( a = 1 \).

(b) [Sketch will be given in class.]

3. (a) \( \lim_{x \to \infty} \cos x \) does not exist (because there is no value \( L \) which the values \( \cos x \) get close to, and stay close to, when \( x \) is large)

(b)

\[
\lim_{x \to \infty} \frac{3x + 5}{x - 4} = \lim_{x \to \infty} \frac{3x + 5}{x - 4} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{3 + \frac{5}{x}}{1 - \frac{4}{x}} = \frac{3 + 0}{1 - 0} = 3
\]

4. [Sketch will be given in class.]