Johnson 1.1. If \( w \in C[0,1] \) and if \( \int_0^1 wv \, dx = 0 \) for all functions \( v \in V = H_0^1 \) then \( w \equiv 0 \) (i.e. \( w(x) = 0 \) for all \( x \in [0,1] \)).

Proof. By contradiction. We suppose \( w \) is not identically zero.

For instance, assume \( A = w(a) > 0 \) for some \( a \in (0,1) \). Since \( w \) is continuous, there is \( \epsilon > 0 \) so that \( w(x) \geq A/2 \) for \( x \in (a-\epsilon,a+\epsilon) \) and so that \( (a-\epsilon,a+\epsilon) \subset [0,1] \). Let \( v \) be the continuous function which equals one in \( (a-\epsilon/2,a+\epsilon/2) \), is zero outside \( (a-\epsilon,a+\epsilon) \), and is linear between. Then

\[
\int_0^1 wv \, dx \geq \int_{a-\epsilon/2}^{a+\epsilon/2} A \frac{h}{2} \, dx = \frac{A}{2} \epsilon > 0,
\]

a contradiction to \( w(a) > 0 \).

We have the essential proof already, but one must also consider the cases \( a = 0 \) and \( a = 1 \) as well as when \( w(a) < 0 \). Once those cases are covered we get a contradiction to \( w \) being nonzero somewhere. \( \square \)

Johnson 1.3. Let \( 0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1 \) be a partition of \( I = [0,1] \). Let \( I_j = [x_{j-1}, x_j] \) for \( j = 1, \ldots, M+1 \). Describe a basis of “hat-like” functions for

\[
V_h = \{ v(x) : v \in C(I), v(0) = v(1) = 0, \text{ and } v \text{ is quadratic on each } I_j \}.
\]

Formulate the finite element method for boundary value problem \( (D) \) on \( V_h \), and fill in the details if the partition is uniform.

Solution. Consider the dimension of \( V_h \). For each quadratic function on an interval \( I_j \) there are three unknown parameters. But the imposition of continuity, justified because we seek \( V_h \subset H_0^1 \), gives one constraint at each of the \( x_j \), \( j = 0, \ldots, M + 1 \). Thus

\[
\dim V_h = 3(M+1) - (M+2) = 2M + 1.
\]

Thus we need \( 2M + 1 \) “hat” functions.

Let \( h_j = x_j - x_{j-1} \). I define

\[
\psi_j(x) = \frac{4(x_j - x)(x - x_{j-1})}{h_j^2}, \quad j = 1, \ldots, M + 1
\]

for \( x \in I_j \) and zero otherwise. Define

\[
\varphi_j(x) = \frac{(x_{j+1} - x)(x - x_{j-1})}{h_{j+1}h_j}, \quad j = 1, \ldots, M
\]

for \( x \in I_{j-1} \cup I_j \) and zero otherwise. Note \( \psi_j(x_k) = 0 \) for all \( k \) and \( \psi_j((x_{j-1} + x_j)/2) = 1 \). On the other hand, \( \varphi_j(x_k) = \delta_{jk} \). It is clear that \( \{ \psi_j, \varphi_j \} \) is a basis of \( V_h \).
It is easy to write the form of the system $A\xi = b$ corresponding to the FEM on $(D)$ using this basis. Let $u_h = \sum_{j=1}^{M+1} \xi_j \psi_j + \sum_{j=M+2}^{2M+1} \xi_j \varphi_{j-M-1}$. Then the fundamental FEM principle

$$(u_h', v') = (f, v) \quad \text{for all } v \in V_h,$$

which we impose by choosing $v$ to be the basis elements, becomes

\[
\begin{align*}
\sum_{j=1}^{M+1} \xi_j (\psi_j', \psi_k') + \sum_{j=M+2}^{2M+1} \xi_j (\varphi_{j-M-1}', \psi_k') &= (f, \psi_k), \\
\sum_{j=1}^{M+1} \xi_j (\varphi_{j-M}', \psi_k') + \sum_{j=M+2}^{2M+1} \xi_j (\varphi_{j-M-1}', \varphi_k') &= (f, \varphi_k),
\end{align*}
\]

for $k = 1, \ldots, M + 1$.

The details of the above system are more calculable for a uniform partition with spacing $h$, but they still take a lot of paper. We might start by calculating that $\psi_j' (x) = \frac{1}{h^2} (-2x + (x_{j-1} + x_j))$ and thus:

$$(\psi_j', \psi_k') = \delta_{jk} \int_{I_j} (\psi_j')^2 \, dx = \frac{\delta_{jk}}{h^4} \int_{x_{j-1}}^{x_j} (-2x + (x_{j+1} + x_j))^2 \, dx = \ldots$$

but my time is limited, too. \hfill \square

**Johnson 1.4.** Formulate a finite difference method for $(D)$ and compare to (1.6).

**Solution.** Let $0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1$ be a uniform partition of $I = [0, 1]$ with $x_j - x_{j-1} = h = \frac{1}{M+1}$. Recall that $u''(x) = \frac{1}{h^2} (u(x+h) - 2u(x) + u(x-h)) + O(h^2)$; see any numerical analysis book and note the $O(h^2)$ error depends on $u^{(iv)}$. Suppose $w_j$ approximates $u(x_j)$ for $j = 1, \ldots, M$. Then $-u'' = f$ is approximated by

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} = f(x_j)$$

for $j = 1, \ldots, M$, as long as we interpret $w_0 = 0$ and $w_{M+1} = 0$, thus incorporating the boundary conditions. As a matrix equation this is

\[
\begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
\vdots \\
w_M
\end{pmatrix}
= \begin{pmatrix}
f(x_1) \\
f(x_2) \\
f(x_3) \\
\vdots \\
f(x_M)
\end{pmatrix}.
\]

Let’s compare to (1.6). By multiplying (1) by $h$, the left sides are identical, but we note that in (1.6) the load elements are $b_j = (f, \varphi_j) = \int_{x_{j-1}}^{x_{j+1}} f(x) \varphi_j(x) \, dx$ while in the finite difference case we have $\tilde{b}_j = hf(x_j)$. It turns out that $b_j, \tilde{b}_j$ are both pretty good, but distinct, approximations to the integral

$$\int_{x_{j-1} - \frac{h}{2}}^{x_{j+1} + \frac{h}{2}} f(x) \, dx,$$
in the finite element case using a weighting (hat) function which extends outside of the interval \([x_j - \frac{h}{2}, x_j + \frac{h}{2}]\) and in the finite difference case by application of the midpoint rule. If \(f(x)\) happens to be linear then the results will be identical; otherwise generally not.