On Defective Ramsey Numbers

Glenn G. Chappell*
John Gimbel

University of Alaska Fairbanks

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Let G be a graph, and let k be a nonnegative integer. A set T of vertices of G is k-*sparse* if the subgraph of G induced by T has maximum degree at most k.

Let G be a graph, and let k be a nonnegative integer. A set T of vertices of G is k-*sparse* if the subgraph of G induced by T has maximum degree at most k.

For nonnegative integers k, a, b, we define $R_k(a,b)$ to be the least n such that for each graph G of order n, either G contains a k-sparse set of a vertices, or $\overline{G}$ contains a k-sparse set of b vertices.

R_k introduced by Ekim & Gimbel (not published yet), generalizing R_1 introduced by Cockayne & Mynhardt (1999).

Example.

$R_1(4,4) = ?$

R_0: ordinary 2-color Ramsey numbers.

R_0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	
4	1	4	9	18	25					
5	1	5	14	25						
6	1	6	18							
7	1	7	23							
8	1	8	28							
9	1	9	36							
10	1	10								

R_1: values due to Cockayne & Mynhardt.

R_1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	2	3	4	5	6	7	8	9	10
4	2	4	6	9	11	16	17		
5	2	5	9	15					
6	2	6	11						
7	2	7	16						
8	2	8	17						
9	2	9							
10	2	10							

R_2 and higher: just one nontrivial value already known. $R_2(5,5) = 7$ (Ekim & Gimbel).

R_2	3	4	5	6	7	8
3	3	3	3	3	3	3
4	3	4	5	6	7	8
5	3	5	**7**			
6	3	6				
7	3	7				
8	3	8				

We have found various new values. Most of these were found using a computer.

$R_2(5,6) = 8$ (proven by hand)
$R_2(5,7) = 10$
$R_2(5,8) = 12$
$R_2(6,6) = 12$
$R_3(6,7) = 9$
$R_3(6,8) = 10$
$R_3(6,9) = 12$
$R_3(7,7) = 11$
$R_3(7,8) = 13$
$R_4(7,8) = 10$
$R_4(7,9) = 11$
$R_4(7,10) = 13$
$R_4(8,8) = 12$
$R_5(8,9) = 11$
$R_5(8,10) = 12$
$R_5(9,9) = 13$
$R_6(9,10) = 12$

We have now determined all values of $R_k(a,b)$ for k, a, b for which this value is at most 12.

Improved tables (bold values are newly established):

R_2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3
4	3	4	5	6	7	8	9	10
5	3	5	7	**8**	**10**	**12**		
6	3	6	**8**	**12**				
7	3	7	**10**					
8	3	8	**12**					
9	3	9						
10	3	10						

R_3	4	5	6	7	8	9	10	11
4	4	4	4	4	4	4	4	4
5	4	5	6	7	8	9	10	11
6	4	6	**8**	**9**	**10**	**12**		
7	4	7	**9**	**11**	**13**			
8	4	8	**10**	**13**				
9	4	9	**12**					
10	4	10						
11	4	11						

Improved tables (cont'd):

R_4	5	6	7	8	9	10	11	12
5	5	5	5	5	5	5	5	5
6	5	6	7	8	9	10	11	12
7	5	7	**9**	**10**	**11**	**13**		
8	5	8	**10**	**12**				
9	5	9	**11**					
10	5	10	**13**					
11	5	11						
12	5	12						

R_5	6	7	8	9	10	11	12	13
6	6	6	6	6	6	6	6	6
7	6	7	8	9	10	11	12	13
8	6	8	**10**	**11**	**12**			
9	6	9	**11**	**13**				
10	6	10	**12**					
11	6	11						
12	6	12						
13	6	13						

What about asymptotic behavior of R_k?

We can generalize a 1947 result of Erdős on ordinary Ramsey numbers.

Theorem. *Let k be a nonnegative integer.*

(a) There exists a constant $t = t(k) > 1$ such that, if $a \geq 2$, then $R_k(a,a) > t^a$.

(b) If $a \geq k+2$, then $R_k(a,a) < 4^{a-k-2}(k+4)$.

- Proof idea for (a). Use Erdős-Gimbel result on subgraphs of a random graph.

- Proof idea for (b). Use Ekim-Gimbel result: $R_k(a,b) \leq R_k(a-1,b) + R_k(a,b-1)$.

Corollary. *For fixed k, the value of* $\log R_k(a, a)$ *is* $\Theta(a)$.

Computing infinite families of values.

Theorem. *Let* $k, a \geq 0$. *Then the following hold.*

(a) If $k \geq a-3$, *then* $R_k(k+a, k+a) \geq k+3a-4$.

(b) If $a \geq 2$ *and* $k \geq 3a-6$, *then*
$R_k(k+a, k+a) = k+3a-4$.

Corollary. *For each pair of integers a, b, there exist constants $\ell_{a,b}$ and $u_{a,b}$ so that*

$$\ell_{a,b} \leq R_k(k+a, k+b) - k \leq u_{a,b}$$

for all $k \geq 0$ for which $R_k(k+a, k+b)$ is defined.

In other words, for fixed a, b, the value of $R_k(k+a, k+b)$ is $k + O(1)$.

What are the values of $R_k(k+a, k+a)$ like, for fixed a?

$R_k(k+3, k+3)$	value	
$R_0(3,3)$	6	
$R_1(4,4)$	6	individual computations
$R_2(5,5)$	7	
$R_3(6,6)$	8	
$R_4(7,7)$	9	infinite family
$R_5(8,8)$	10	

$R_k(k+4, k+4)$	value	
$R_0(4,4)$	18	
$R_1(5,5)$	15	
$R_2(6,6)$	12	individual computations
$R_3(7,7)$	11	
$R_4(8,8)$	12	
$R_5(9,9)$	13	
$R_6(10,10)$	14	
$R_7(11,11)$	15	
$R_8(12,12)$	16	infinite family
$R_9(13,13)$	17	

Proposition. *For all $k \geq 0$, we have the following.*

(a) $R_k(k+3, k+3) = \max\{6, 5+k\}$.

(b) $R_k(k+4, k+4) = \max\{18-3k, 8+k\}$.

Does this behavior continue, for larger values of k?

And just what is "this behavior" anyway?

Paper, slides, and—coming soon—source code (Python):

`http://www.cs.uaf.edu/~chappell/papers/defram`