On Defective Ramsey Numbers (DRAFT)

Glenn G. Chappell
Department of Computer Science
University of Alaska
Fairbanks, AK 99775-6670
chappellg@member.ams.org

John Gimbel
Department of Mathematics and Statistics
University of Alaska
Fairbanks, AK 99775-6660
jggimbel@alaska.edu

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Abstract

If T is a set of vertices of a graph G, then T is k-sparse in G if the subgraph of G induced by T has maximum degree at most k. Following Ekim & Gimbel [4], we define generalized Ramsey numbers: $R_k(a, b)$, for nonnegative integers k, a, b, is the least n such that, for each graph G of order n, either G contains a k-sparse set of a vertices, or the complement of G contains a k-sparse set of b vertices. We study R_k, proving basic properties and bounds.

We compute various values of R_k. We show that, if $a \geq 2$ and $k \geq 3a - 6$, then $R_k(k + a, k + a) = k + 3a - 4$. We compute other specific values of $R_k(a, b)$, some using a computer. In particular, we determine $R_k(a, b)$ for all k, a, b for which this value is at most 12.

We also analyze certain asymptotic behaviors of R_k. We show that, for fixed a, b, the value of $R_k(k + a, k + b)$ is $k + O(1)$. We further show that, for fixed k, the value of $\log R_k(a, a)$ is $\Theta(a)$.

1 Introduction

Let k be a nonnegative integer. Given a (finite, undirected) graph G, a set T of vertices of G is *k-sparse* in G if the subgraph of G induced by T has maximum degree at most k. Some authors refer to a k-sparse set as "k-dependent". A 0-sparse set is the same as an independent set.

Following Ekim & Gimbel [4] we define generalized Ramsey numbers: $R_k(a, b)$ is the least n such that, for each graph G of order n, either G contains a k-sparse set of a vertices, or $\overline{G}$ contains a k-sparse set of b vertices. Note that that values of R_0 are the usual 2-color Ramsey numbers.

Note that the function R_k can be thought of in a graph Ramsey number context. If $\mathcal{A}$, $\mathcal{B}$ are sets of graphs, then $R(\mathcal{A}, \mathcal{B})$ is the least n such that, for each graph G of order n, either G contains a subgraph isomorphic to an element of $\mathcal{A}$, or $\overline{G}$ contains a subgraph isomorphic to an element of $\mathcal{B}$. Say a graph H is *k-dense* if $V(H)$ is k-sparse in $\overline{H}$. Let $\mathcal{A}$ be the set of all k-dense graphs on a vertices, and let $\mathcal{B}$ be the set of all k-dense graphs on b vertices. It is not hard to see that $R_k(a, b) = R(\mathcal{A}, \mathcal{B})$.

Thus, when we find values of R_k, we are also determining more traditional graph Ramsey numbers.

Such reasoning has been used, for example, by Cockayne & Mynhardt [3, Cor. 3(iii)], to determine $R_1(5, 5)$. The 4-spoke wheel, W_4, is 1-dense. Further, every 1-dense graph of order 5 has a subgraph isomorphic to W_4. Thus, $R_1(5, 5) = R(W_4, W_4)$. Cockayne & Mynhardt reference Harborth & Mengersen [9, Thm. 2], who showed that $R(W_4, W_4) = 15$. (That $R(W_4, W_4) = 15$ was also stated without proof by Hendry [10]; see Radziszowski [11, Sect. 4.2].)

In this paper, we study R_k. In Section 2, we list previously known values of R_k. In Section 3, we give basic properties and bounds on R_k. In Section 4, we analyze the behavior of $R_k(k+a, k+b)$, when a, b are fixed and k increases. In Section 5, we compute various values of R_k, including nontrivial infinite families of values, as well as some values determined using a computer. In Section 6, we continue our discussion of asymptotic behavior of R_k. We turn our attention to $R_k(a, a)$ when k is fixed and a increases.

For a graph G, we denote the vertex set of G by $V(G)$. If $T \subseteq V(G)$, then $G[T]$ is the subgraph of G induced by T.

2 Previously Known Values

The following table shows the known values of $R_0(a, b)$—that is, ordinary 2-color Ramsey numbers—for $1 \le a, b \le 11$. See the survey by Radziszowski [11, Sect. 2.1]. We use the obvious facts that $R_0(1, b) = 1$ and $R_0(2, b) = b$; see Lemma 3.1(f) and (g).

R_0	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11
3	1	3	6	9	14	18	23	28	36		
4	1	4	9	18	25						
5	1	5	14	25							
6	1	6	18								
7	1	7	23								
8	1	8	28								
9	1	9	36								
10	1	10									
11	1	11									

The following table shows the known values of $R_1(a,b)$, for $2 \le a,b \le 10$. These are from Cockayne & Mynhardt [3]; also see Ekim & Gimbel [4]. We also use the facts that $R_1(2,b) = 2$ and $R_1(3,b) = b$; see Lemma 3.1(f) and (g).

R_1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	2	3	4	5	6	7	8	9	10
4	2	4	6	9	11	16	17		
5	2	5	9	15					
6	2	6	11						
7	2	7	16						
8	2	8	17						
9	2	9							
10	2	10							

The following table shows the previously known values of $R_2(a,b)$, for $3 \le a,b \le 7$. Of these, one nontrivial value was known before this work: $R_2(5,5) = 7$, from Ekim & Gimbel [4, Thm. 3]. We also use the facts that $R_2(3,b) = 3$ and $R_2(4,b) = b$; see Lemma 3.1(f) and (g).

R_2	3	4	5	6	7	8
3	3	3	3	3	3	3
4	3	4	5	6	7	8
5	3	5	7			
6	3	6				
7	3	7				
8	3	8				

In Section 5 we will add to the above table.

3 Basic Properties

The following lemma gives basic properties of k-sparseness and R_k. Some parts of the lemma—(b), (e), (f), and special cases of (g)—were observed by Ekim & Gimbel [4, Remarks 2, 3, 5–7] and Cockayne & Mynhardt [3, Prop. 1, Cor. 3(i)].

Lemma 3.1. *Let k, a, and b be nonnegative integers. Then the following all hold.*

(a) Let G be a graph, and let $T \subseteq V(G)$ with $|T| \le k+1$. Then T is k-sparse in G.

(b) Let G be a graph, and let $T \subseteq V(G)$ with $|T| = k+2$. Then either T is k-sparse in G, or T is k-sparse in $\overline{G}$.

(c) Let G be a graph, and let $T \subseteq V(G)$. Then T is k-sparse in G iff every $(k+2)$-subset of T is k-sparse in G.

(d) $R_{k+1}(a,b) \le R_k(a,b)$.

(e) $R_k(a,b) = R_k(b,a)$.

(f) If $a \le k+1$ or $b \le k+1$, then $R_k(a,b) = \min\{a,b\}$.

(g) $R_k(k+2,b) = b$.

Proof. (a) This is obvious.

(b) If T is not k-sparse in G, then some $x \in T$ is adjacent to $k+1$ other vertices of T, that is, to all other vertices of T. Thus, in the subgraph of $\overline{G}$ induced by T, x has degree 0, and every other vertex of T has degree at most k, since each such vertex is not adjacent to x. Therefore, T is k-sparse in $\overline{G}$.

(c) Clearly, if T is k-sparse in G, then every $(k+2)$-subset of T is k-sparse in G.

If T is not k-sparse in G, then some $x \in T$ is adjacent to at least $k+1$ other vertices of T. Let $U \subseteq T$ consist of x and $k+1$ of its neighbors. Then U is a $(k+2)$-subset of T that is not k-sparse in G.

(d) This follows from the fact that every k-sparse set is also $(k+1)$-sparse.

(e) This is obvious.

(f) This follows from part (a).

(g) If $b \le k+1$, then the result follows from part (f). Therefore, suppose that $b \ge k+2$.

Consider K_{b-1}. This graph does not contain a k-sparse set of order $k+2$. Furthermore, since its order is less than b, there can be no set of b vertices that is k-sparse in the complement. Thus, $R_k(k+2,b) \ge b$.

Now let G be a graph of order b containing no k-sparse set of order $k+2$. By part (b) every $(k+2)$-vertex subset of $V(G)$ is k-sparse in $\overline{G}$. Therefore, by part (c), $V(G)$ is a b-vertex set that is k-sparse in $\overline{G}$, and so $R_k(k+2,b) \le b$. □

The following lemma gives simple bounds for R_k. Part (b) generalizes a result of Burr, Erdős, Faudree, & Shelp [1, Thm. 2], who proved it for $k = 0$. Part (c) was observed by Ekim & Gimbel [4, Remark 4].

Lemma 3.2. *Let k, a, b, c be nonnegative integers. Then the following hold.*

(a) If $a \ge 1$ and $b \ge k+2$, then $R_k(a,b) \ge R_k(a-1,b)+1$.

(b) If $a \ge 2k+1$ and $b,c \ge 1$, then $R_k(a,b+c-1) \ge R_k(a,b)+R_k(a,c)-1$.

(c) If $a,b \ge 1$, then $R_k(a,b) \le R_k(a-1,b)+R_k(a,b-1)$.

Proof. (a) If $a = 1$, then the statement follows from Lemma 3.1(f).

Suppose $a \ge 2$. Let $n = R_k(a-1,b)-1$. Note that $n \ge 0$. Let G be a graph of order n, such that G contains no $(a-1)$-vertex k-sparse set, and $\overline{G}$ contains no b-vertex k-sparse set.

Let G^* be G with an additional isolated vertex x added. Then G^* has order $n+1 = R_k(a-1,b)$, and G^* contains no a-vertex k-sparse set. If $n < k+1$, then, since $b \ge k+2$, the graph $\overline{G^*}$ has order less than b, and so it can contain no b-vertex k-sparse set. On the other hand, if $n \ge k+1$, then adding x to some $(b-1)$-vertex k-sparse set in $\overline{G}$ results in a set inducing a subgraph in which x has degree greater than k. Thus $\overline{G^*}$ contains no b-vertex k-sparse set.

We conclude that $R_k(a,b)$ is greater than the order of G^*; the statement follows.

(b) Let G_1 be a graph of order $R_k(a,b)-1$ such that G_1 has no k-sparse a-set, and $\overline{G_1}$ has no k-sparse b-set. Similarly, let G_2 be a graph of order $R_k(a,c)-1$ such that G_2 has no k-sparse a-set, and $\overline{G_2}$ has no k-sparse c-set. Let G be the graph formed by taking the disjoint union of G_1 and G_2 and adding all edges between vertices in G_1 and vertices in G_2.

Graph G has order $R_k(a,b)+R_k(a,c)-2$. We claim that G has no k-sparse a-set. To see this, let $S \subseteq V(G)$ with $|S| = a$. If S lies entirely in either G_1 or G_2, then S is not k-sparse. Thus, since $a \ge 2k+1$, set S must contain at least $k+1$ vertices of either G_1 or G_2, and it must contain a vertex v in the other G_i. This vertex v thus has degree at least $k+1$ in the subgraph of G induced by S. We see that S is not k-sparse.

Further, $\overline{G}$ has no k-sparse $(b+c-1)$-set, since any $(b+c-1)$-set in $V(G)$ must contain either b vertices of $\overline{G_1}$ or c vertices of $\overline{G_2}$, in which case it is not k-sparse in $\overline{G}$.

We conclude that $R_k(a,b+c-1)$ is greater than the order of G; the statement follows.

(c) Let $n = R_k(a-1,b)+R_k(a,b-1)$, and let G be a graph of order n. Let $x \in V(G)$. Then either x has at least $R_k(a-1,b)$ non-neighbors or x has at least $R_k(a,b-1)$ neighbors. We consider the former case; the other is handled similarly.

Let T be the set of non-neighbors of x. If T has a b-vertex subset that is k-sparse in $\overline{G}$, then we are done. Otherwise, T must have an $(a-1)$-vertex subset U that is k-sparse in G. Then $U \cup \{x\}$ is an a-vertex set that is k-sparse in G. □

It seems likely that part (b) of Lemma 3.2 holds for smaller values of a, perhaps for $a \ge k+2$.

4 Asymptotic Behavior I

In the following proposition, we use Lemma 3.2 to establish bounds on $R_k(k+a, k+b)$ in terms of k, a, and b.

Proposition 4.1. *Let $k \geq 0$, and let $a, b \geq 2$. Then the following hold.*

(a) $R_k(k+a, k+b) \geq k+a+b-2$.

(b) $R_k(k+a, k+b) \leq \binom{a+b-4}{a-2}k + \binom{a+b-2}{a-1}$.

Proof. (a) We proceed by induction on a. In the base case, $a = 2$. We need to show that $R_k(k+2, k+b) \geq k+b$. This follows from Lemma 3.1(g).

If $a > 2$, then we apply Lemma 3.2(a) to obtain

$$\begin{aligned} R_k(k+a, k+b) &\geq R_k(k+a-1, k+b) + 1 \\ &\geq (k+a+b-3) + 1 \\ &= k+a+b-2. \end{aligned}$$

(b) We proceed by induction, first on a, and then on b. If $a = 2$, then the right-hand side of the inequality equals $k + b$, and we need to show that $R_k(k+2, k+b) \leq k+b$. This follows from Lemma 3.1(g). The inequality similarly holds when $b = 2$.

Now assume that $a, b \geq 3$, and that the inequality holds for all smaller values of a and, with the given value of a, for all smaller values of b. Apply Lemma 3.2(c) to obtain

$$\begin{aligned} R_k(k+a, k+b) &\leq R_k(k+a-1, k+b) + R_k(k+a, k+b-1) \\ &\leq \left[\binom{a+b-5}{a-3}k + \binom{a+b-3}{a-2}\right] + \left[\binom{a+b-5}{a-2}k + \binom{a+b-3}{a-1}\right] \\ &= \binom{a+b-4}{a-2}k + \binom{a+b-2}{a-1}. \quad \square \end{aligned}$$

Proposition 4.1 implies that, for fixed a, b, the value of $R_k(k+a, k+b)$ is $\Theta(k)$. We will prove a stronger statement: that this value is $k + O(1)$—thus showing that part (b) of Proposition 4.1 is far from best possible. We begin by finding exact formulas for $R_k(k+a, k+a)$ when k is sufficiently large.

Theorem 4.2. *Let $k, a \geq 0$. Then the following hold.*

(a) If $k \geq a-3$, then $R_k(k+a, k+a) \geq k+3a-4$.

(b) If $a \geq 2$ and $k \geq 3a-6$, then $R_k(k+a, k+a) = k+3a-4$.

Proof. (a) If $a = 0, 1$, then the statement follows from Lemma 3.1(f). If $a = 2$, then the statement follows from Lemma 3.1(g).

Suppose that $a \geq 3$ and $k \geq a-3$. Define a graph $D_{k,a}$ as follows. Let P, Q, R be disjoint sets of vertices with $|P| = |Q| = 2a-4$ and $|R| = k-(a-3)$. Let the vertex set of $D_{k,a}$ be $P \cup Q \cup R$. Add edges: let the edges between sets P, Q form a regular bipartite graph with degree $a-2$. Let each vertex of Q be adjacent to every other vertex of Q and every vertex of R. This defines $D_{k,a}$. Note that $P \cup R$ is an independent set in $D_{k,a}$, while Q induces a complete subgraph.

$D_{k,a}$ has order $(2a-4)+(2a-4)+[k-(a-3)] = k+3a-5$. Thus, to obtain a set of $k+a$ vertices of $D_{k,a}$, we would remove $2a-5$ vertices.

Let $S \subseteq V(D_{k,a})$ with $|S| = k+a$. The set Q contains $2a-4$ vertices. Thus, S contains at least 1 vertex of Q. Each vertex in Q has degree $(a-2)+(2a-5)+[k-(a-3)] = k+2a-4$. Thus, the subgraph of $D_{k,a}$ induced by S has a vertex of degree at least $(k+2a-4)-(2a-5) = k+1$, and so S cannot be k-sparse in $D_{k,a}$.

Similarly, the set P contains $2a-4$ vertices. Thus, S contains at least 1 vertex of P. Each vertex in P has degree $a-2$. Thus, the subgraph of $D_{k,a}$ induced by S has a vertex of degree at most $a-2$, which has degree at least $(k+a-1)-(a-2) = k+1$ in $\overline{D_{k,a}}$. Hence, S cannot be k-sparse in $\overline{D_{k,a}}$.

We see that $D_{k,a}$ is a graph of order $k+3a-5$ such that neither $D_{k,a}$ nor its complement has a k-sparse set of $k+a$ vertices. Statement (a) follows.

(b) Because $a \geq 2$ and $k \geq 3a-6$, we have $k \geq a-3$, and so we can apply part (a). It remains to show that $R_k(k+a, k+a) \leq k+3a-4$. Suppose for a contradiction that this is false. Then there must exist a graph G with order $k+3a-4$, such that each $(k+a)$-vertex subset of $V(G)$ is k-sparse in neither G nor $\overline{G}$. That is, each $(k+a)$-vertex subset of $V(G)$ induces a subgraph of G having a vertex of degree at least $k+1$ and a vertex of degree at most $a-2 = (k+a-1)-(k+1)$.

We say a vertex v is *strong* in G if there exists some $(k+a)$-vertex induced subgraph of G in which v has degree at least $k+1$. Thus v is strong in G iff the degree of v in G is at least $k+1$.

We say a vertex v is *weak* in G if there exists some $(k+a)$-vertex induced subgraph of G in which v has degree at most $a-2$. Thus v is weak in G iff the degree of v in G is at most $3a-6 = (a-2)+[(k+3a-4)-(k+a)]$.

Note that $k+1 > 3a-6$, and so no vertex can be both strong and weak in G. (*Note: This is why we need* $k \geq 3a-6$.)

There must exist at least $2a-3$ weak vertices, since, otherwise, we can remove $2a-4$ vertices (noting that $2a-4 \geq 0$, since $a \geq 2$), leaving a set of $k+a$ vertices, none of which is weak in G. Such a set would be k-sparse in $\overline{G}$.

We say a vertex v that is weak in G is *special* if v is adjacent to at most $a-2$ strong vertices in G. If we remove $2a-4$ weak vertices from G, then the resulting induced subgraph has order $k+a$, and so must contain a vertex x of degree at most $a-2$. Since we only removed weak vertices, and no weak vertex is strong, the subgraph must contain every strong vertex of G, and so x is a special weak vertex. Since we can remove any collection of $2a-4$ weak vertices of G and find a special weak vertex in what remains, G must contain at least $2a-3$ special weak vertices.

Let $S \subseteq V(G)$ be a set of $2a-3$ special weak vertices. Let $T \subseteq V(G)$ be the set of all strong vertices of G that are adjacent to more than $a-2$ vertices of S. Note that S, T are disjoint. Because each vertex in S is adjacent to at most $a-2$ strong vertices, we have $|T| < |S| = 2a - 3$, and so $|V(G) - T| \geq k + a$.

Let U be a set of $k + a$ vertices of G, such that $S \subseteq U \subseteq V(G) - T$. Such a set U exists, because $a - 3 \leq k$, and so $|S| = 2a - 3 \leq k + a = |U|$. We claim that this U is k-sparse (which would be a contradiction). To see this, consider a vertex $z \in U$. If z is not strong in G, then z has degree at most k. If z is strong in G, then, since $z \notin T$, z is adjacent to at most $a - 2$ vertices of S. There are $(k + a) - (2a - 3) - 1 = k - a + 2$ vertices of U, other than z, that do not lie in S. Thus, in the subgraph of G induced by U, vertex z has degree at most $(a - 2) + (k - a + 2) = k$. We see that U is k-sparse.

By contradiction, statement (b) is proven. □

Using Theorem 4.2, we can show that, for fixed a, b, the value of $R_k(k + a, k + b)$ is $k + O(1)$.

Corollary 4.3. *For each pair of integers a, b, there exist constants $\ell_{a,b}$ and $u_{a,b}$ so that*

$$\ell_{a,b} \leq R_k(k + a, k + b) - k \leq u_{a,b}$$

for all $k \geq 0$ for which $R_k(k + a, k + b)$ is defined.

Proof. Fix integers a, b. Without loss of generality, say $a \geq b$. If $b < 2$, then the result follows from Lemma 3.1(f), with $\ell_{a,b} = u_{a,b} = b$.

Suppose that $b \geq 2$; then $a \geq 2$ as well. Let $\ell_{a,b}$, $u_{a,b}$ be defined as follows.

$$\begin{aligned} \ell_{a,b} &= a + b - 2; \\ u_{a,b} &= \max_{0 \leq k \leq 3a-6} \left[R_k(k + a, k + a) - k \right]. \end{aligned}$$

The lower bound now follows from Proposition 4.1(a). We consider the upper bound. Note that $u_{a,b}$ is well defined, since we take the maximum value of a nonempty finite set.

By Lemma 3.2(a), since $a \geq b$, we have $R_k(k + a, k + b) \leq R_k(k + a, k + a)$. It thus suffices to show that $R_k(k + a, k + a) - k \leq u_{a,b}$. When $k \leq 3a - 6$ this follows from the definition of $u_{a,b}$. If $k > 3a - 6$, then we have

$$\begin{aligned} R_k(k + a, k + a) - k &= 3a - 4 && \text{by Theorem 4.2(b)} \\ &= R_{3a-6}([3a - 6] + a, [3a - 6] + a) - [3a - 6] && \text{by Theorem 4.2(b)} \\ &\leq u_{a,b}. \quad \square \end{aligned}$$

It appears that, for fixed $a, b \geq 0$, the value $R_k(k + a, k + b) - k$ is maximized when $k = 0$, and thus that we can set $u_{a,b} = R(a, b)$ in Corollary 4.3.

Conjecture 4.4. *If $k, a, b \geq 0$, then $R_k(k + a, k + b) - k \leq R(a, b)$.* □

Conjecture 4.4 would follow from the following stronger conjecture.

Conjecture 4.5. *For fixed integers a, b, the sequence of values of $R_k(k + a, k + b) - k$ is nonincreasing.* □

We will discuss asymptotic behavior again later, in Section 6, after we determine a number of previously unknown values of R_k.

5 Specific Values

Using Theorem 4.2, we can establish, for each k, the first nontrivial value of R_k. That is, we find the first value that is not given by Lemma 3.1.

Corollary 5.1. *Let $k \geq 0$. Then,*

$$R_k(k+3, k+3) = \begin{cases} 6, & \text{if } k = 0, \\ k+5, & \text{otherwise.} \end{cases}$$

Proof. When $k = 0$ we use the well known result that $R(3,3) = 6$ (noted by Greenwood & Gleason [8, p. 3]). The case $k = 1$ was proven by Cockayne & Mynhardt [3, Cor. 3(ii)]. The case $k = 2$ was proven by Ekim & Gimbel [4, Thm. 3].

When $k \geq 3$, we set $a = 3$, note that $k \geq 3a - 6$, and apply Theorem 4.2(b). □

Now we determine a number of previously unknown individual values of $R_k(a, b)$. We will give the full proof for one value: $R_2(5, 6) = 8$. For the others, we give proofs for the lower bounds; the upper bounds were verified using a computer.

Theorem 5.2. $R_2(5, 6) = 8$.

Proof. For convenience, we will actually prove that $R_2(6, 5) = 8$. The lower bound follows from Lemma 3.2(a) and the fact that $R_2(5, 5) = 7$ (proven by Ekim & Gimbel [4, Thm. 3]).

For the upper bound, suppose for a contradiction that there exists a graph G of order 8, such that there is no 2-sparse set of order 6 in G, and there is no 2-sparse set of order 5 in $\overline{G}$. We note that G can contain neither a 5-cycle nor $K_{2,3}$ as a subgraph (not necessarily induced), for otherwise $\overline{G}$ would contain a 2-sparse set of order 5.

Maximum Degree at Most 3—We claim that G has maximum degree at most 3.

Suppose for a contradiction that G has a vertex v of degree at least 4. Let $S \subseteq V(G)$ be a set of 4 vertices that belong to the open neighborhood of v. Let $T = V(G) - [S \cup \{v\}]$; note that $|T| = 3$. The S-*degree* of a vertex that does not lie in S, is defined to be the cardinality of the intersection of its open neighborhood with S. Say $T = \{x, y, z\}$, with the S-degree of x being at least that of y, which, in turn, is at least that of z.

As G does not contain a 5-cycle, we see that $G[S]$ cannot contain a path on four vertices. As G does not contain $K_{2,3}$, we see that S is 2-sparse. Thus, $G[S]$ is isomorphic to a subgraph of either $K_3 \cup K_1$ or $K_2 \cup K_2$.

S**-Degree at Most** 1**.** We wish to show that, for S, T defined above, each vertex in T has S-degree at most 1. If x has S degree 3 or more, then G contains a $K_{2,3}$. We may thus assume that every vertex in T has S-degree at most 2.

Suppose that x has S-degree exactly 2. Then the 2 neighbors of x in S might be adjacent, but cannot be adjacent to other vertices of S, for otherwise G would contain a 5-cycle. In particular, S must be 1-sparse in G.

Suppose that y also has S-degree exactly 2. Then, as G contains no 5-cycle, x and y must be nonadjacent. As G does not contain a $K_{2,3}$, we see that x and y cannot have exactly the same neighborhood in S. If x and y have a common neighbor in S, then S is an independent set, and $S \cup \{x, y\}$ forms a 2-sparse set of order six. On the other hand, if x and y each have S-degree 2, but share no common neighbor, then $S \cup \{x, y\}$ induces a subgraph of two disjoint triangles, and hence is 2-sparse. We see that y has S-degree at most 1.

If y and z have a common neighbor $w \in S$, then neither y nor z can be adjacent to x, since G contains no 5-cycle, and so $(S \cup T) - \{w\}$ is a 2-sparse 6-set. On the other hand, if there is no such w, then $S \cup \{y, z\}$ is a 2-sparse 6-set.

Thus, we have shown that each vertex of T has S-degree at most 1.

Finishing the Maximum Degree 3 Proof. We now complete the verification of our claim that G has maximum degree at most 3. Recall that $G[S]$ is isomorphic to a subgraph of either $K_3 \cup K_1$ or $K_2 \cup K_2$.

We wish to show, first, that there is at most 1 vertices in $G[S \cup T]$ with degree at least 4, and, second, that if 2 vertices in $G[s \cup T]$ have degree at least 3, then they are adjacent.

For the first part, note that S, T are each 2-sparse. Thus, any vertex lying in one of these sets and having degree at least 4 in $G[S \cup T]$, must be adjacent to at least 2 vertices in the other set. Since there are at most 3 edges between S, T, there can be only 1 such vertex.

For the second part, let a, b be vertices of degree at least 3 in $G[S \cup T]$. Suppose that $a \in T$. If $b \in T$, then $G[T]$ is K_3, and so a, b are adjacent. On the other hand, if $b \in S$, then a must be adjacent to both other vertices of T. Since G contains no 5-cycle, there can be only one vertex in S that is adjacent to a vertex of T. This vertex must thus be b, and so a, b are adjacent.

Now suppose that $a, b \in S$. Then one of the two has degree at least 2 in $G[S]$, while the other has degree at least 1. Considering the possible isomorphism classes of $G[S]$, we see that a, b must be adjacent.

The first and second parts, above, having been verified, we conclude that removing a vertex of maximum degree from $G[S \cup T]$ leaves a 2-sparse set of 6 vertices.

Thus, our claim holds: G has maximum degree at most 3.

Triangle Free—We claim that G is triangle-tree.

Suppose for a contradiction that G contains a triangle. Let N be the set of vertices that do not lie in the triangle, and have at least one neighbor in the triangle. Because G has maximum degree at most 3, each vertex in the triangle has at most 1 neighbor in N, and so $|N| \leq 3$. If $|N| \leq 2$, then the vertices of the triangle together with 3 other vertices that do not lie in N, form a 2-sparse 6-set. Thus $|N| = 3$, and so there is a matching between the triangle and N.

Let u, v be the 2 vertices of G in neither the triangle nor in N. As G contains no 5-cycle, neither u nor v can have more than 1 neighbor in N, and $G[N]$ can have no edges. If u and v have a common neighbor, say w, then the removal of the neighbor of w in the triangle leaves a 2-sparse set. On the other hand, if u and v do not share a common neighbor, then the removal of any vertex in the triangle leaves a 2-sparse set.

Thus, our claim holds: G is triangle-free.

Handling a Bipartite Graph—Suppose that G is not bipartite. Then G contains an induced odd cycle. As this cycle can be neither a triangle nor a 5-cycle, it must be an induced 7-cycle, which is 2-sparse. We may thus assume that G is bipartite.

Let A, B be the partite sets of G, where $|A| \leq |B|$. As B is 2-sparse, we must have $|B| \leq 5$. Accordingly, $|B| \in \{4, 5\}$.

We first consider the case $|B| = 4$.

Suppose that both A and B contain a vertex of degree 3. If these 2 vertices are nonadjacent, then the removal of both vertices leaves a 2-sparse 6-set. Thus, each vertex of degree 3 in A must be adjacent to each vertex of degree 3 in B. The removal of one such vertex from A and one from B leaves a 2-sparse 6-set. We may thus assume, without loss of generality, that A contains no vertices of degree 3.

The set B cannot contain 3 vertices of degree 3, since these would necessarily have a common neighbor, which would be a vertex of degree 3 in A. Hence, we may remove 2 vertices of B to obtain a 2-sparse 6-set.

In our final case, we have $|B| = 5$, and hence $|A| = 3$.

If B contains at least 2 vertices of degree 3, then G contains a $K_{2,3}$. If B contains exactly 1 vertex of degree 3, then the removal of this vertex leaves a 2-sparse set. Thus, B contains no vertices of degree 3. If at most 2 vertices of A have degree 3, then we remove them and produce a 2-sparse set. We may thus assume that all vertices of A have degree 3. Hence some vertex of B must have degree 2. Remove this vertex and its nonneighbor in A; what remains is a 2-sparse set of 6 vertices.

This exhausts all cases. Thus, no such G exists; our desired conclusion follows. □

Using a computer program, we have determined other values of R_k. Our software is written in the Python programming language; it is available via the Worldwide Web [2].

We have also been able to enumerate the number of extremal graphs for these values of R_k. A graph G is *extremal* for $R_k(a, b)$ if G has order $R_k(a, b) - 1$, G contains no k-sparse set of a vertices, and $\overline{G}$ contains no k-sparse set of b vertices. Informally, G is extremal if its existence shows that $R_k(a, b)$ is at least its actual value.

Proposition 5.3. *The following all hold.*

(a) $R_2(5, 7) = 10$, *with exactly* 16 *extremal graphs.*

(b) $R_2(5, 8) = 12$, *with exactly* 8 *extremal graphs.*

(c) $R_2(6, 6) = 12$, *with exactly* 2 *extremal graphs.*

(d) $R_3(6, 7) = 9$, *with exactly* 28 *extremal graphs.*

(e) $R_3(6, 8) = 10$, *with exactly* 159 *extremal graphs.*

(f) $R_3(6, 9) = 12$, *with exactly* 4 *extremal graphs.*

(g) $R_3(7, 7) = 11$, *with exactly* 4 *extremal graphs.*

(h) $R_3(7,8) = 13$*, with exactly* 43 *extremal graphs.*

(i) $R_4(7,8) = 10$*, with exactly* 84 *extremal graphs.*

(j) $R_4(7,9) = 11$*, with exactly* 550 *extremal graphs.*

(k) $R_4(7,10) = 13$*, with exactly* 4 *extremal graphs.*

(l) $R_4(8,8) = 12$*, with exactly* 8 *extremal graphs.*

(m) $R_5(8,9) = 11$*, with exactly* 316 *extremal graphs.*

(n) $R_5(8,10) = 12$*, with exactly* 2430 *extremal graphs.*

(o) $R_5(9,9) = 13$*, with exactly* 22 *extremal graphs.*

(p) $R_6(9,10) = 12$*, with exactly* 1712 *extremal graphs.*

The upper bounds were all verified using a computer program [2]. We give proofs for the lower bounds.

Proof of Lower Bounds. (a) For the lower bound, we can use the following 9-vertex graph G, which is extremal for $R_2(7,5)$. Begin with a 6-cycle. Let S be an independent set of 3 vertices in this cycle. For each $v \in S$, add a new vertex v' having the same neighbors as v. Let G be the resulting graph.

Then G has no 7-vertex 2-sparse set and $\overline{G}$ has no 5-vertex 2-sparse set, showing that $R_2(7,5) > 9$.

(b) For the lower bound, we can use the following 11-vertex graph G, which is extremal for $R_2(8,5)$. The vertex set of G is $\{1,2,3,a,b,c,d,w,x,y,z\}$, with edges as follows. Vertices a,b,c,d induce a K_4. Vertices w,x,y,z induce a K_4. Vertex 1 is adjacent to a and w. Vertex 2 is adjacent to b and x. Vertex 3 is adjacent to c and y.

Then G has no 8-vertex 2-sparse set and $\overline{G}$ has no 5-vertex 2-sparse set, showing that $R_2(8,5) > 11$.

(c) For the lower bound, we can use the following 11-vertex graph G, which is extremal for $R_2(6,6)$. The vertex set of G is $\{1,2,3,4,a,b,c,d,t,x,y\}$, with edges as follows. Vertices $1,a,2,b,3,c,4,d$ form an 8-cycle, in that order. The set $\{a,b,c,d,t\}$ induces a K_5. Each vertex of $\{1,2,3,4\}$ is adjacent to each vertex of $\{x,y\}$, and t is adjacent to y.

Then G has no 6-vertex 2-sparse set and $\overline{G}$ has no 6-vertex 2-sparse set, showing that $R_2(6,6) > 11$.

(d) The lower bound follows from Lemma 3.2(a) and the fact that $R_3(6,6) = 8$, by Theorem 4.2(b).

(e) The lower bound follows from Lemma 3.2(a) and the fact that $R_3(6,7) = 9$, from part (d).

(f) For the lower bound, we can use the following 11-vertex graph G, which is extremal for $R_3(9,6)$. The vertex set of G is $\{1,2,3,4,5,6,a,b,c,x,y\}$, with edges as follows. Vertices

a, b, c, x, y induce a K_5. Vertices $1, 2$ are each adjacent to a. Vertices $3, 4$ are each adjacent to b. Vertices $5, 6$ are each adjacent to c.

Then G has no 9-vertex 3-sparse set and $\overline{G}$ has no 6-vertex 3-sparse set, showing that $R_3(9, 6) > 11$.

(g) The lower bound follows from Theorem 4.2(a).

(h) For the lower bound, we can use the following 12-vertex graph G, which is extremal for $R_3(8, 7)$. The vertex set of G is $\{1, 2, 3, 4, 5, a, b, c, d, e, f, x\}$, with edges as follows. Vertices a, b, c, d, e, f induce a K_6. Vertices 1, 2 are each adjacent to a and b. Vertices 3, 4 are each adjacent to c and d. Vertex 5 is adjacent to e and f. Vertex x is adjacent to 1, 2, 3, 4, 5, and 6.

Then G has no 8-vertex 3-sparse set and $\overline{G}$ has no 7-vertex 3-sparse set, showing that $R_3(8, 7) > 12$.

(i) The lower bound follows from Lemma 3.2(a) and the fact that $R_4(7, 7) = 9$, by Theorem 4.2(b).

(j) The lower bound follows from Lemma 3.2(a) and the fact that $R_4(7, 8) = 10$, from part (i).

(k) For the lower bound, we can use the following 12-vertex graph G, which is extremal for $R_4(10, 7)$. The vertex set of G is $\{1, 2, 3, 4, 5, 6, a, b, c, x, y, z\}$, with edges as follows. Vertices a, b, c, x, y, z induce a K_6. Vertices 1, 2 are each adjacent to a. Vertices 3, 4 are each adjacent to b. Vertices 5, 6 are each adjacent to c.

Then G has no 10-vertex 4-sparse set and $\overline{G}$ has no 7-vertex 4-sparse set, showing that $R_4(10, 7) > 12$.

(l) The lower bound follows from Theorem 4.2(a).

(m) The lower bound follows from Lemma 3.2(a) and the fact that $R_5(8, 8) = 10$, by Theorem 4.2(b).

(n) The lower bound follows from Lemma 3.2(a) and the fact that $R_5(8, 9) = 11$, from part (m).

(o) The lower bound follows from Theorem 4.2(a).

(p) The lower bound follows from Lemma 3.2(a) and the fact that $R_6(9, 9) = 11$, by Theorem 4.2(b). □

Remark 5.4. *Using our computer program [2], we determined that there are exactly* 13 *extremal graphs for* $R_2(5, 6)$. □

We can now update our tables of values of R_k. Note that we have computed no new values of R_0 or R_1.

The following table shows the values of $R_2(a, b)$ that we have found, for $3 \le a, b \le 10$. These are from Lemma 3.1(f) and (g), Ekim & Gimbel [4, Thm. 3] (for $R_2(5, 5) = 7$), Theorem 5.2, and Proposition 5.3. Newly established values are shown in boldface.

R_2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3
4	3	4	5	6	7	8	9	10
5	3	5	7	**8**	**10**	**12**		
6	3	6	**8**	**12**				
7	3	7	**10**					
8	3	8	**12**					
9	3	9						
10	3	10						

The following table shows the values of $R_3(a, b)$ that we have found, for $4 \leq a, b \leq 11$. These are from Lemma 3.1(f) and (g), Theorem 4.2(b), and Proposition 5.3. Again, newly established values are shown in boldface.

R_3	4	5	6	7	8	9	10	11
4	4	4	4	4	4	4	4	4
5	4	5	6	7	8	9	10	11
6	4	6	**8**	**9**	**10**	**12**		
7	4	7	**9**	**11**	**13**			
8	4	8	**10**	**13**				
9	4	9	**12**					
10	4	10						
11	4	11						

The following table shows the values of $R_4(a, b)$ that we have found, for $5 \leq a, b \leq 12$. These are from Lemma 3.1(f) and (g), Theorem 4.2(b), and Proposition 5.3.

R_4	5	6	7	8	9	10	11	12
5	5	5	5	5	5	5	5	5
6	5	6	7	8	9	10	11	12
7	5	7	**9**	**10**	**11**	**13**		
8	5	8	**10**	**12**				
9	5	9	**11**					
10	5	10	**13**					
11	5	11						
12	5	12						

The following table shows the values of $R_5(a, b)$ that we have found, for $6 \leq a, b \leq 13$. These are from Lemma 3.1(f) and (g), Theorem 4.2(b), and Proposition 5.3.

R_5	6	7	8	9	10	11	12	13
6	6	6	6	6	6	6	6	6
7	6	7	8	9	10	11	12	13
8	6	8	**10**	**11**	**12**			
9	6	9	**11**	**13**				
10	6	10	**12**					
11	6	11						
12	6	12						
13	6	13						

The following table shows the values of $R_6(a, b)$ that we have found, for $7 \le a, b \le 14$. These are from Lemma 3.1(f) and (g), Theorem 4.2(b), and Proposition 5.3.

R_6	7	8	9	10	11	12	13	14
7	7	7	7	7	7	7	7	7
8	7	8	9	10	11	12	13	14
9	7	9	**11**	**12**				
10	7	10	**12**	**14**				
11	7	11						
12	7	12						
13	7	13						
14	7	14						

The following table shows the values of $R_7(a, b)$ that we have found, for $8 \le a, b \le 15$. These are from Lemma 3.1(f) and (g), and Theorem 4.2(b).

R_7	8	9	10	11	12	13	14	15
8	8	8	8	8	8	8	8	8
9	8	9	10	11	12	13	14	15
10	8	10	**12**					
11	8	11		**15**				
12	8	12						
13	8	13						
14	8	14						
15	8	15						

Remark 5.5. *We have determined $R_k(a, b)$ for all k, a, b for which $R_k(a, b) \le 12$. These are the values corresponding to entries in the above tables that are at most* 12*, along with other values that can easily be computed using Lemma 3.1(f) and (g).* □

6 Asymptotic Behavior II

Once again, we are interested in the behavior of $R_k(k + a, k + b)$ when a, b are fixed and k increases. We know the following, from Corollary 5.1.

$R_k(k+3,k+3)$	value
$R_0(3,3)$	6
$R_1(4,4)$	6
$R_2(5,5)$	7
$R_3(6,6)$	8
$R_4(7,7)$	9
$R_5(8,8)$	10

The values below are from Greenwood & Gleason [8, p. 4] (for $R_0(4,4) = 18$), Cockayne & Mynhardt [3, Cor. 3(iii)] (for $R_1(5,5) = 15$), Proposition 5.3, and Theorem 4.2.

$R_k(k+4,k+4)$	value
$R_0(4,4)$	18
$R_1(5,5)$	15
$R_2(6,6)$	12
$R_3(7,7)$	11
$R_4(8,8)$	12
$R_5(9,9)$	13
$R_6(10,10)$	14
$R_7(11,11)$	15
$R_8(12,12)$	16
$R_9(13,13)$	17

More generally, we have the following.

Proposition 6.1. *For all $k \geq 0$, we have the following.*

(a) $R_k(k+3,k+3) = \max\{6, 5+k\}$.

(b) $R_k(k+4,k+4) = \max\{18-3k, 8+k\}$. □

In both cases above, the Ramsey number is the maximum of two polynomials of degree at most 1 in k. Based on this, we indulge in wild speculation: does this continue to be true for other $R_k(k+a,k+a)$? For other $R_k(k+a,k+b)$?

It appears that, for fixed a, there is a unique k_a such that the values $R_k(k+a,k+a)$ are nonincreasing for $k \leq k_a$, and increasing for $k \geq k_a$. For example, we have $k_3 = 1$ and $k_4 = 3$. We ask about the behavior of this k_a.

Question 6.2. *Does this value k_a exist for each a? If so, what is the behavior of k_a as a grows?* □

We have discussed the behavior of $R_k(k+a,k+b)$ when a, b are fixed and k grows large. What about when k is fixed and a, b increase? We establish bounds for the diagonal values $R_k(a,a)$. We will make use of the following theorem due to Erdős & Gimbel [6, Thm. 3]. (Note that a statement *almost surely* holds, if the probability of it holding converges to 1—in this case, as $n \to \infty$.)

Theorem 6.3 (Erdős & Gimbel 1991 [6, Thm. 3])**.** *Given a fixed graph H and a random graph G of order n, the largest H-free subgraph of G almost surely has cardinality less than $c \ln n$ where c is dependent only on H.* □

The following theorem generalizes a result of Erdős [5, Thm. 1], who proved it for $k = 0$ (with $t = \sqrt{2}$ for $a \geq 3$). (Erdős attributes the special case of part (b) when $k = 0$ to G. Szekeres, citing a paper of Erdős & Szekeres [7].)

Theorem 6.4. *Let k be a nonnegative integer.*

(a) There exists a constant $t = t(k) > 1$ such that, if $a \geq 2$, then $R_k(a, a) > t^a$.

(b) If $a \geq k + 2$, then $R_k(a, a) < 4^{a-k-2}(k + 4)$.

Proof. (a) Let H be the graph formed by the disjoint union of $K_{1,k+1}$ and K_1. Let c be that given by Theorem 6.3 for this H. Let $n = \lfloor e^{a/c} \rfloor$. By Theorem 6.3, if n is sufficiently large, then there exists a graph G of order n such that every subset of $V(G)$ with cardinality at least $c \ln n$ induces a subgraph of G containing a copy of H; thus, every subset of cardinality at least a induces such a subgraph. By definition of H, this subgraph is k-sparse in neither G nor $\overline{G}$, and so $R_k(a, a) > n$. Thus, $R_k(a, a) > \left(e^{1/c}\right)^a$, when $n = \lfloor e^{a/c} \rfloor$ is sufficiently large.

We have verified the statement for sufficiently large a, since, if a is large, then n is large. We can verify the statement for all $a \geq 2$ using reasoning similar to that in the proof of the upper bound in Corollary 4.3. Let a_0 be the least "sufficiently large" value of a, or 2 if this value is less than 2. Let t_0 be defined as follows.

$$t_0 = \min_{2 \leq a \leq a_0} \left[\left(R_k(a, a) - \frac{1}{2} \right)^{1/a} \right].$$

Note that this is well defined, since, first, for $a \geq 2$ we have $R_k(a, a) \geq 2$, and so the number being raised to a power is greater than 1, while the exponent is positive, and, second, t_0 is the minimum value of a nonempty finite set.

Lastly, we set $t = \min \left\{ t_0, e^{1/c} \right\}$. We can see that, for this t, we have $R_k(a, a) > t^a$ for all $a \geq 2$.

(b) We can apply Proposition 4.1(b) to show that

$$R_k(a, a) \leq \binom{2a - 2k - 4}{a - k - 2} k + \binom{2a - 2k - 2}{a - k - 1}.$$

The desired statement then follows from the fact that $\binom{2s}{s} < 4^s$ when $s \geq 1$ (this bound can be proven using a simple inductive argument). □

We see that, for fixed k, the values of $R_k(a, a)$ grow exponentially (and thus the values of $R_k(k + a, k + a)$ do as well).

Corollary 6.5. *For fixed k, the value of $\log R_k(a, a)$ is $\Theta(a)$.* □

References

[1] S. A. Burr, P. Erdős, R. J. Faudree, and R. H. Schelp, On the difference between consecutive Ramsey numbers, *Utilitas Math.* **35** (1989), 115–118.

[2] G. G. Chappell, ramseyk [computer program], March 4, 2011, available at <http://www.cs.uaf.edu/~chappell/papers/defram>.

[3] E. J. Cockayne and C. M. Mynhardt, On 1-dependent Ramsey numbers for graphs, *Discuss. Math. Graph Theory* **19** (1999), no. 1, 93–110.

[4] T. Ekim and J. Gimbel, Some defective parameters in graphs, manuscript, 2010.

[5] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[6] P. Erdős and J. Gimbel, A note on the largest H-free subgraph in a random graph, *Graph theory, combinatorics, and applications*, v. 1 (Kalamazoo, MI, 1988), 435–437, Wiley-Intersci. Publ., Wiley, New York, 1991.

[7] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[8] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1–7.

[9] H. Harborth and I. Mengersen, All Ramsey numbers for five vertices and seven or eight edges, *Discrete Math.* **73** (1989), no. 1-2, 91–98.

[10] G. R. T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory* **13** (1989), 245–248.

[11] S. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* (2009), Dynamic Survey 1, revision #12: August 4, 2009, 72 pp. (electronic).