Other Graph Topics
Course Wrap-Up

CS 311 Data Structures and Algorithms

Lecture Slides

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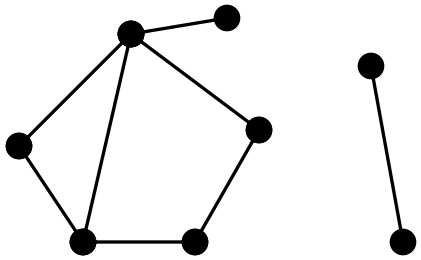
Review

The Rest of the Course Overview

Two Final Topics

- ✓ External Data
  - Previously, we dealt only with data stored in memory.
  - Suppose, instead, that we wish to deal with data stored on an external device, accessed via a relatively slow connection and available in sizable chunks (data on a disk, for example).
  - How does this affect the design of algorithms and data structures?
- (part) Graph Algorithms
  - A **graph** models relationships between pairs of objects.
  - This is a very general notion. Algorithms for graphs often have very general applicability.

This usage of "graph" has nothing to do with the graph of a function. It is a different definition of the word.

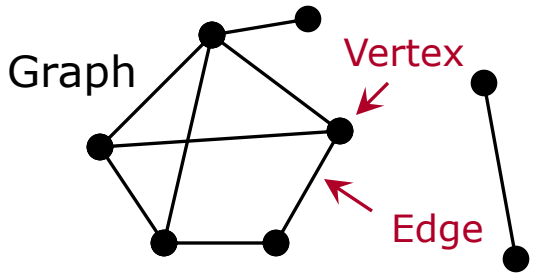


Drawing of a Graph

Review
Introduction to Graphs [1/3]

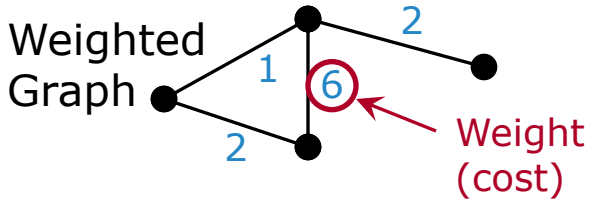
A **graph** consists of **vertices** and **edges**.

- An edge joins two vertices: its **endpoints**.
- 1 **vertex**, 2 vertices (Latin plural).
- Two vertices joined by an edge are **adjacent**; each is a **neighbor** of the other.



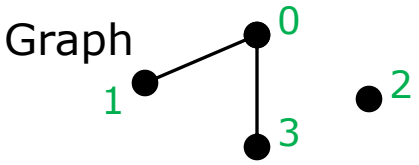
In a **weighted graph**, each edge has a **weight** (or **cost**).

- The weight is the resource expenditure required to use that edge.
- We typically choose edges to minimize the total weight of some kind of collection.



Review
Introduction to Graphs [2/3]

Two common ways to represent graphs.



Adjacency matrix. 2-D array of 0/1 values.

- "Are vertices *i*, *j* adjacent?" in Θ(1) time.
- Finding all neighbors of a vertex is slow for large, sparse graphs.

Adjacency Matrix

	0	1	2	3
0	0	1	0	1
1	1	0	0	0
2	0	0	0	0
3	1	0	0	0

Adjacency lists. List of lists (arrays?).
List *i* holds neighbors of vertex *i*.

- "Are vertices *i*, *j* adjacent?" in Θ(log *N*) time if lists are sorted arrays; Θ(*N*) if not.
- Finding all neighbors can be faster.

N: the number of vertices.

Adjacency Lists

0: 1, 3
1: 0
2:
3: 0

Introduction to Graphs [3/3]

When we analyze the efficiency of graph algorithms, we consider *both* the number of vertices and the number of edges.

- N = number of vertices
- M = number of edges

When analyzing efficiency, we consider adjacency matrices & adjacency lists separately.

The *total* size of the input is:

- For an adjacency matrix: N^2. So $\Theta(N^2)$.
- For adjacency lists: $N + 2M$. So $\Theta(N + M)$.

Some particular algorithm might have order (say) $\Theta(N + M \log N)$.

Review
Graph Traversals [1/2]

To **traverse** a graph means to visit each vertex once.

Two kinds of graph traversals.

- **Depth-first search** (**DFS**)
- **Breadth-first search** (**BFS**)

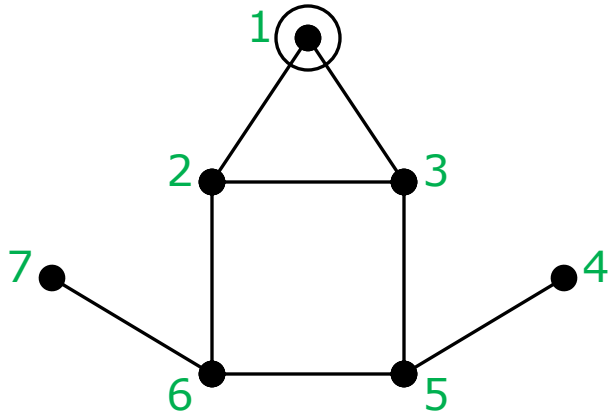
DFS has natural formulations in recursive form and also in non-recursive form, using a Stack.

If, in the non-recursive form of DFS, we turn the Stack into a Queue, we get a BFS algorithm.

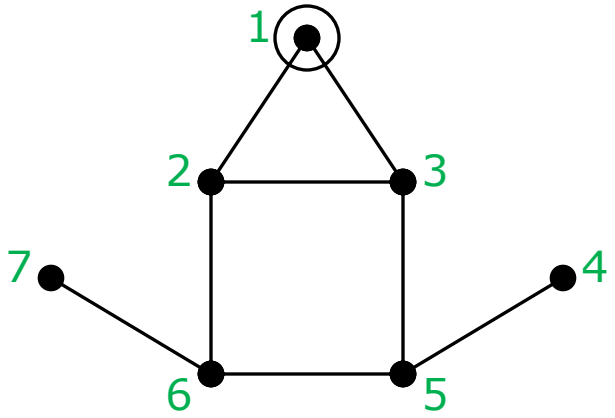
All of these are $\Theta(N + M)$ when given adjacency lists and $\Theta(N^2)$ when given an adjacency matrix. So their running time is of the same order as the total size of the input.

Review
Graph Traversals [2/2]

A DFS and a BFS are illustrated for the graph below.



DFS: …



BFS: …

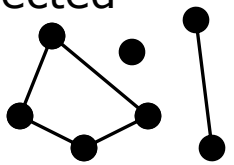
Spanning Trees — Introduction

A **tree** is a graph that:

- Is **connected** (all one piece).
- Has no **cycles**.

Here, "tree" does *not* mean "rooted tree".

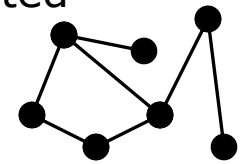
Disconnected Graph



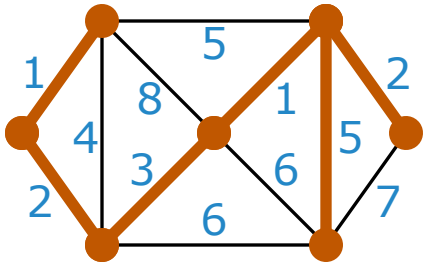
A **spanning tree** in a graph *G* is a tree that:

- Includes only vertices and edges of *G*.
- Includes *all* vertices of *G*.

Connected Graph



Fact. Every connected graph has a spanning tree.



An important problem: given a weighted graph, find a **minimum spanning tree**—a spanning tree of minimum total weight.

There are several nice algorithms that solve this problem.

Spanning Trees — Greedy Methods

A **greedy** method is "shortsighted". It proceeds in a series of choices, each based on *what is known at the time*. Choices are:

- **Feasible**: each makes sense.
- **Locally optimal**: best possible based on current information.
- **Irrevocable**: once a choice is made, it is permanent.

Being greedy is usually *not* a good way to get correct answers.

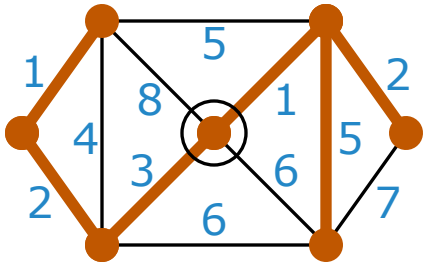
However, in the cases when being greedy gives correct results, it tends to be *very fast*.

Spanning Trees — Prim's Algorithm [1/4]

Prim's Algorithm is a greedy algorithm to find a minimum spanning tree in a connected weighted graph.

Idea

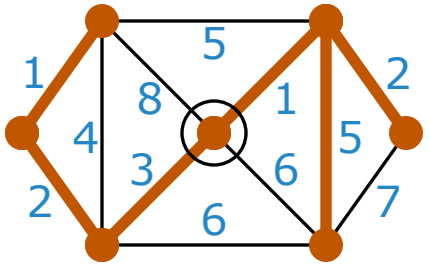
- One vertex is specified as *start*.
- As the algorithm proceeds, we add edges to a tree. Using these edges, we are able to reach more and more vertices from *start*.
- Procedure. Repeatedly add the lowest-weight edge from a reachable vertex to a non-reachable vertex, until all vertices are reachable.



Spanning Trees — Prim's Algorithm [2/4]

Prim's Algorithm

- Given: Connected graph, weights on the edges; one vertex is *start*.
- Returns: Edge set of a minimum spanning tree.
- Procedure:
  - Mark all vertices as not-reachable.
  - Set edge set of tree to empty.
  - Mark *start* vertex as reachable.
  - Repeat while there exist not-reachable vertices:
    - Find lowest weight edge joining a reachable vertex to a not-reachable vertex.
    - Add this edge to the tree.
    - Mark the not-reachable endpoint of this edge as reachable.
  - Return edge set of tree.



It is not obvious that Prim's Algorithm correctly finds a minimum spanning tree. *But it does!*

Spanning Trees — Prim's Algorithm [3/4]

Finding the lowest-weight edge from reachable to not-reachable:

- Use a Priority Queue holding edges, ordered by weight and implemented as a Minheap. So we do getFront & delete on the edge of *least* weight.
- Insert edges that join reachable & not-reachable vertices: when marking a vertex as reachable, insert into the PQ all edges from this vertex to not-reachable neighbors.
- When getting an edge from the PQ, check to be sure it *still* joins reachable & not-reachable vertices. If not, skip it.

Represent a weighted graph as usual + matrix of edge weights.

Easy optimization: stop when the tree has $N-1$ edges.

Spanning Trees — Prim's Algorithm [4/4]

We looked at an implementation of Prim's Algorithms that uses `std::priority_queue` to find the lowest-weight edge.

See `prim.cpp`.

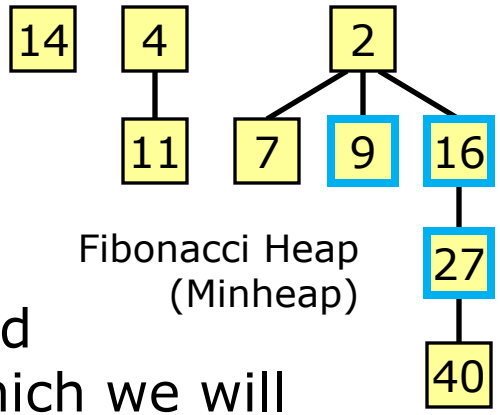
What is the order of our implementation of Prim's Algorithm?

- It does something with each vertex.
- It does something with each edge.
- The latter may involve insertion & deletion in a Binary Heap.

Result: $\Theta(N + M \log M)$.

For a connected graph, we have $M \geq N - 1$.

So: $\Theta(M \log M)$.



Fibonacci Heap
(Minheap)

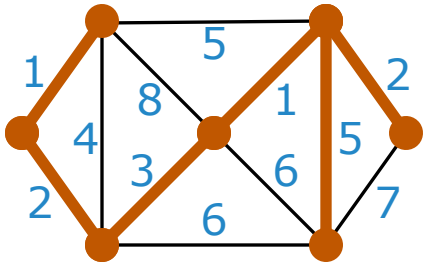
Or: store vertices in the PQ instead of edges, and implement the PQ using a *Fibonacci Heap* (which we will not cover). This is more efficient for graphs with many edges.

Spanning Trees — Kruskal's Algorithm

Another greedy algorithm to find a minimum spanning tree:
Kruskal's Algorithm [J. Kruskal 1956].

Procedure

- Set edge set of tree to empty.
- Repeat:
  - Add the least-weight edge joining two vertices that cannot be reached from each other using edges added so far.
- Return edge set of tree.



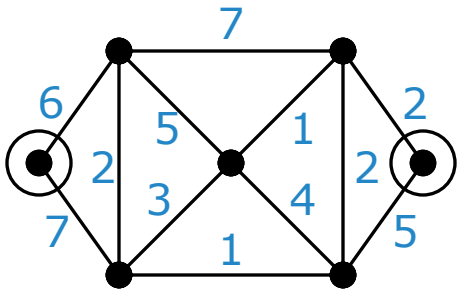
To implement Kruskal's Algorithm well, we need an efficient way to check whether a vertex can be reached from another vertex.

We cover a solution to this problem soon!

Other Graph Topics

Shortest Path [1/3]

Shortest Path Problem. Given a weighted graph with two vertices specified, find the shortest path from one to the other (the path with the lowest total weight).



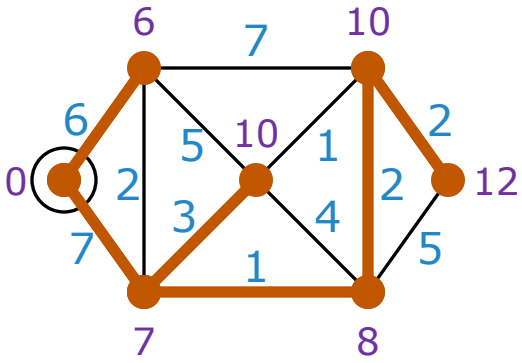
Shortest Path [2/3]

Dijkstra's Algorithm [E. Dijkstra 1959]. Given a *start* vertex, this algorithm finds, for each other vertex in the weighted graph, the shortest path from *start* to that vertex.

Vertex label = length of shortest path from *start* to this vertex that has been found *so far*.

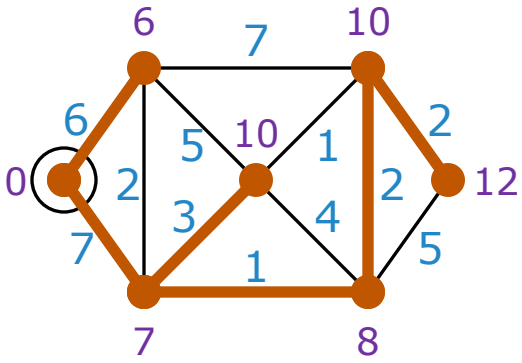
Procedure:

- Label each vertex. *Start* gets 0. Others get ∞.
- Mark all vertices as unvisited.
- Repeat:
  - Find the unvisited vertex with the smallest label. Call it *x*.
  - Mark *x* as visited.
  - For each unvisited neighbor of *x*:
    - Let *newLabel* = (label of *x*) + (weight of edge from *x* to neighbor).
    - If *newLabel* < (label of neighbor), then set (label of neighbor) = *newLabel*.
- Done. Meaning of each vertex label: length of shortest path from *start* to this vertex, or ∞ if there is no path.



Other Graph Topics
Shortest Path [3/3]

Efficiency issues for Dijkstra's Algorithm are similar to those for Prim's Algorithm.



We put vertices in a Heap and allow for decrease-key.

Then, for a connected graph, using a Binary Heap gives $\Theta(M \log N)$, while a Fibonacci Heap gives $\Theta(M + N \log N)$, just as for Prim's Algorithm.

Other Graph Topics
Union-Find [1/7]

Some graph operations do not require a fully general graph representation.

An example of this is given by the operations in ADT called **Union-Find** (a.k.a. **Find-Merge**).

Data

- A graph.

Operations

- **MakeSet**. Create a new vertex.
- **Union**. Given two vertices, add an edge between them.
- **Find**. Given a vertex, determine which **component** (connected chunk) of the graph it lies in.

Using *Find*, we can determine whether there is a path joining two vertices: this is true if they lie in the same component.

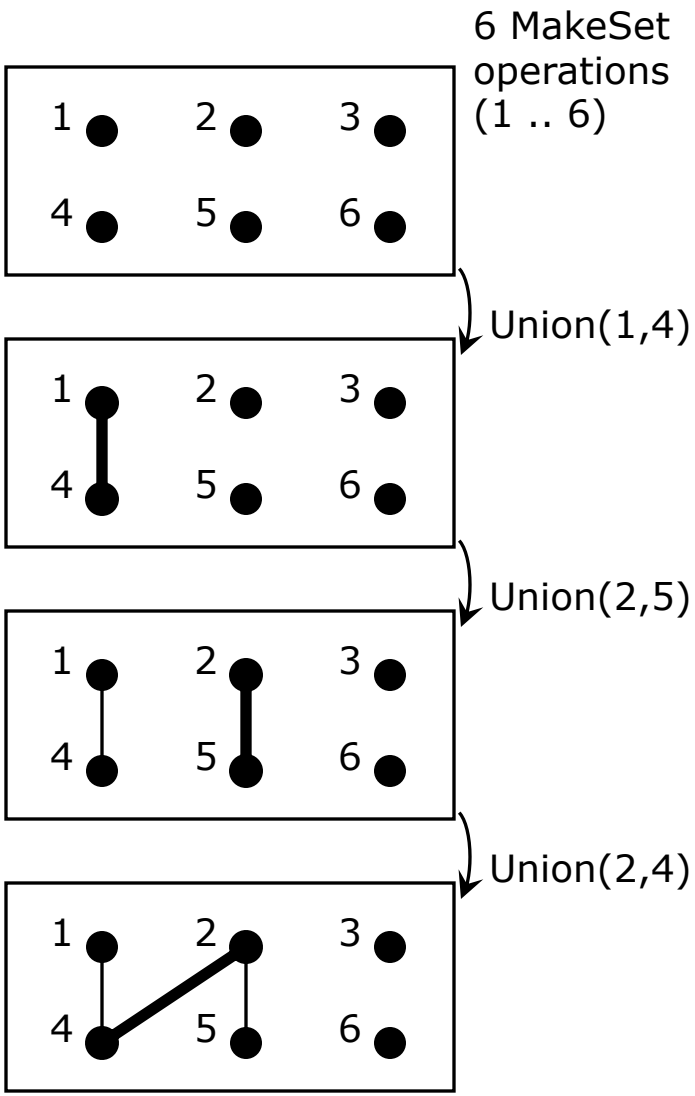
The names of these operations should make more sense shortly.

Union-Find [2/7]

Union-Find operations:

- **MakeSet**. Create a new vertex.
- **Union**. Given two vertices, add an edge between them.
- **Find**. Given a vertex, determine which component of the graph it lies in.

But Union-Find is not really about *graphs* …



Other Graph Topics
Union-Find [3/7]

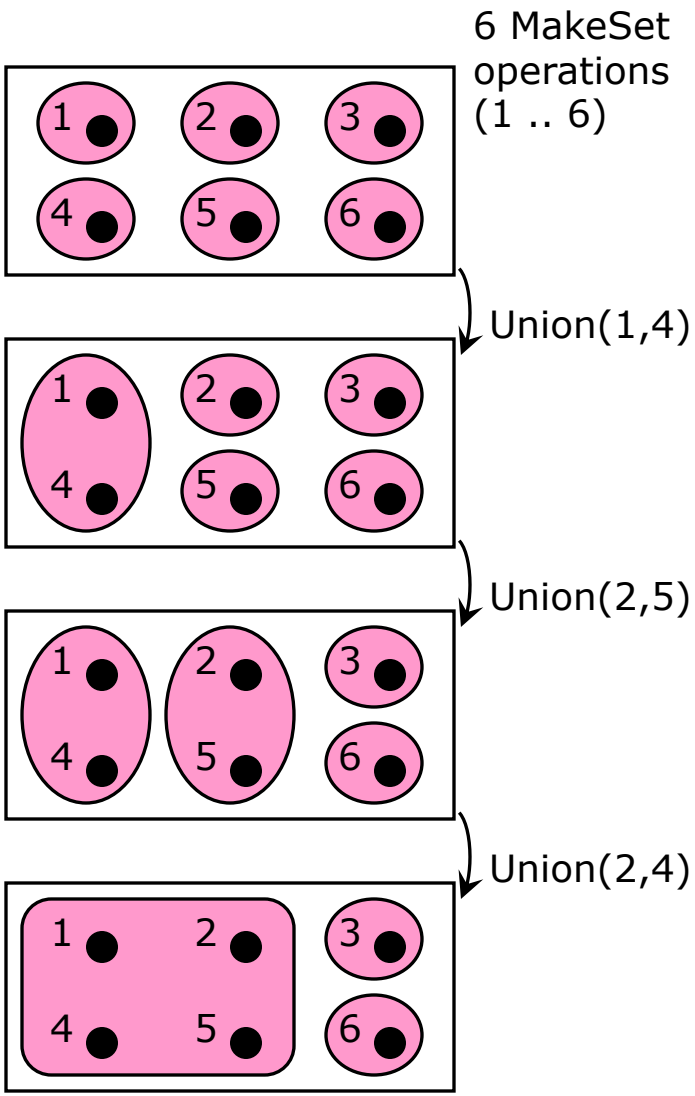
Union-Find is not really about *graphs*; it is about *blobs**.

- Each vertex lies in some blob.
- **MakeSet** creates a new single-vertex blob.
- **Union** merges the blobs containing two given vertices into one blob.
- **Find** determines which blob a given vertex lies in.

Since we only care about blobs, we do not need to keep track of edges.

We must keep track of *something*—but what? We answer this shortly.

Blobs* are actually called **sets. Thus the names of the operations.



Union-Find [4/7]

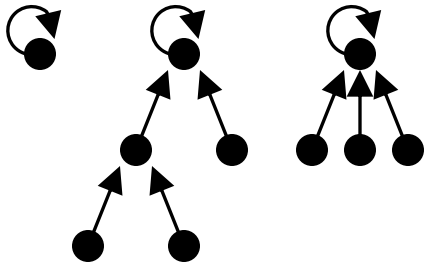
Union-Find operations usually do not use a graph representation.

Rather, the are done on some kind of **disjoint-set structure**.

- Also called a *union-find structure* or *find-merge structure*.

A commonly used—and very efficient—example is a **Disjoint-Set Forest** [B. Galler & M. Fischer 1964].

- Node-based. Each node represents one vertex.
- A set (blob) forms a rooted tree.
- Each node has a pointer. This points to its parent, or to itself if it is the root of a tree.



Doing the Union-Find Operations

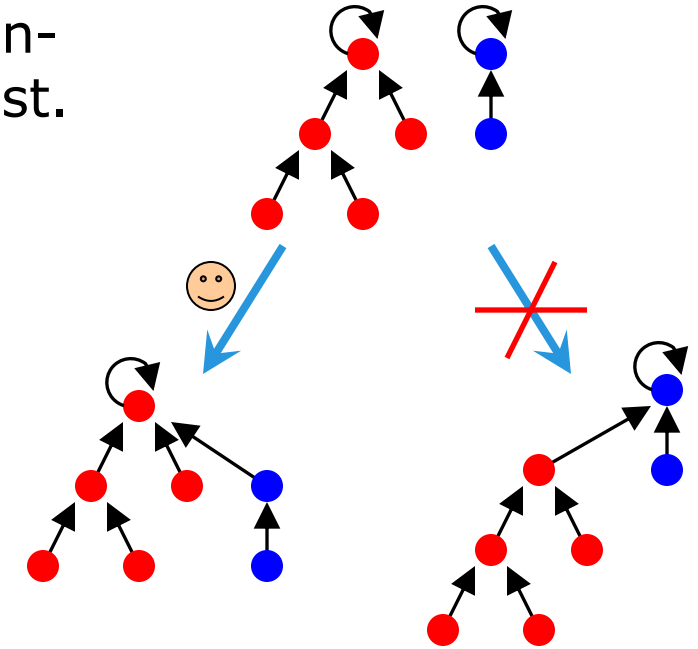
- **MakeSet**. Create a new node, pointing at itself.
- **Find**. Start at the given vertex, and follow the pointer chain to the root. Return identifier for this root—perhaps its index in an array?
  - So the root—or its index—serves as the identifier for a set.
- **Union**. Do a *Find* on each of the two given vertices. Point one root at the other, forming a single tree.

Other Graph Topics
Union-Find [5/7]

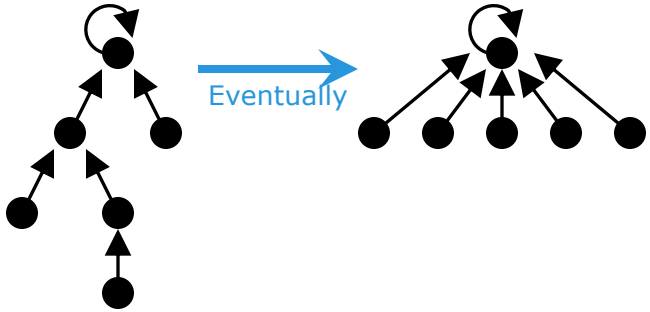
Two optimizations greatly speed up Union-Find operations on a Disjoint-Set Forest.

Union by Rank. When doing a *Union*, if the two trees have different heights, then attach the tree with *smaller* height to the root of the other tree.

- To make this efficient, track the height of each tree in its root.



Path Compression. After following a pointer chain—which happens during a *Find*—point each node at the root.

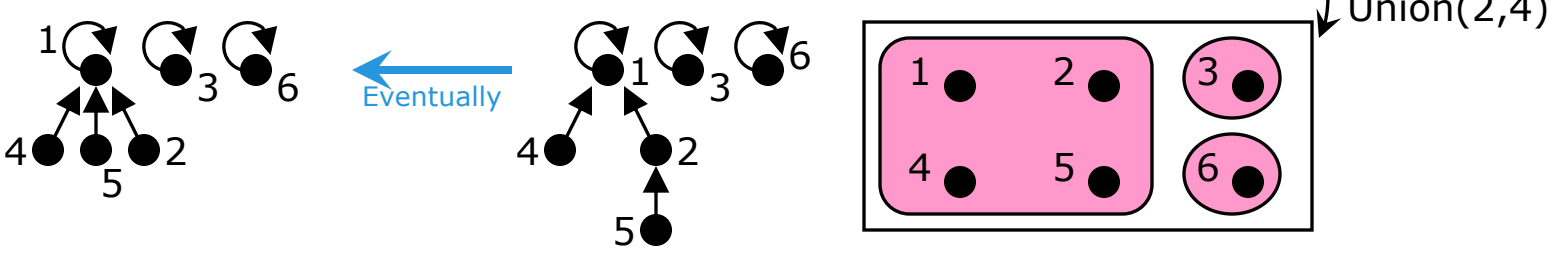


Union-Find [6/7]

MakeSet. Create node pointing at itself.

Union. 2 *Finds*. Point root of small-height tree at other root.

Find. Follow pointer chain to the root. Point each vertex in chain at the root. Return the root.



Union-Find [7/7]

Amortized time per operation for a Disjoint-Set Forest, with the two optimizations discussed, is known to be $O(\alpha(n))$, where $\alpha(n)$ is the *extremely* slow-growing **inverse Ackermann function**.

- A web search finds info about this easily.

Thus: About as near to amortized constant time as one can get, without actually being amortized constant time.

Implementations of Kruskal's Algorithm typically use a Disjoint-Set Forest.

The Rest of the Course
Overview

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  - Suppose, instead, that we wish to deal with data stored on an external device, accessed via a relatively slow connection and available in sizable chunks (data on a disk, for example).
  - How does this affect the design of algorithms and data structures?
- ✓ Graph Algorithms
  - A **graph** models relationships between pairs of objects.
  - This is a very general notion. Algorithms for graphs often have very general applicability.

DONE

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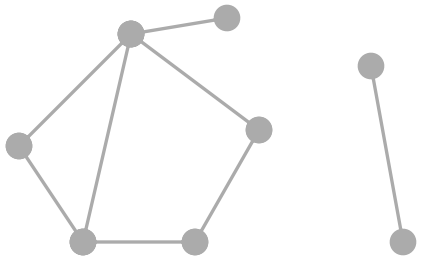

Drawing of a Graph

Course Wrap-Up

Course Wrap-Up
From the First Day of Class: Course Overview — Goals

Upon successful completion of CS 311, you should:

- Have experience writing and documenting high-quality code.
- Understand proper error handling, enabling software components to support robust, reliable applications.
- Be able to perform basic analyses of algorithmic efficiency, including use of big-*O* and related notation.
- Be familiar with various standard algorithms, including those for searching and sorting.
- Understand what data abstraction is, and how it relates to software design.
- Be familiar with standard container data structures, including implementations and relevant trade-offs.

Course Wrap-Up
From the First Day of Class: Course Overview — Topics

The following topics will be covered, roughly in order:

- Advanced C++
- Software Engineering Concepts
- Recursion
- Searching
- Algorithmic Efficiency
- Sorting
- Data Abstraction
- Basic Abstract Data Types / Data Structures:
  - Smart Arrays & Strings
  - Linked Lists
  - Stacks & Queues
  - Trees (various kinds)
  - Priority Queues
  - Tables
- Briefly: external data, graph algorithms.

DONE

Goal: Practical generic containers

A **container** is a data structure holding multiple items, usually all the same type.

A **generic** container is one that can hold objects of client-specified type.

Course Wrap-Up
Things That Matter [1/3]

Scalability *matters!*

The big winners in the modern world are those who design and produce scalable systems.

Remember, we generally look for:

- Constant or logarithmic time for single-item operations.
- Linear or log-linear time (or faster) for whole-dataset operations.

Are we talking about worst-case behavior? Is average-case acceptable, with worst-case being slower? What about amortized time?

Answer. It depends on the requirements of the project.

We measure time by counting basic operations.

These operations may vary. **It matters what we count!**

Course Wrap-Up
Things That Matter [2/3]

Trade-offs *matter!*

Why not just give everyone a quick-reference sheet listing the best solutions to various problems?

Because many problems have no single best overall solution.

Understand trade-offs; find solutions that meet your needs.

Some examples from this semester:

- Finding: Sequential Search vs. Binary Search.
- Sequences: Array vs. Linked List vs. Sequence in a Table, etc.
- Finding again: Binary Search on sorted array vs. Table.
- Tables: Red-Black Tree vs. Hash Table vs. Prefix Tree.
  - And then there is the effect of slow connections (external methods).
- Taking time (& money!) to optimize vs. slow code.
- Using already written data structure/algorithm vs. writing our own.

Course Wrap-Up
Things That Matter [3/3]

Error handling, robustness, and reliability *matter!*

Software systems manage sensitive personal data, financial transactions, military equipment, and medical devices. They drive cars, planes, trains, and ships. They run security systems.

The days when we could get away with saying, "Aw, that hardly ever happens; don't worry about it," are largely gone.

- Responsibilities like those above require very low failure rates.
- The increasing size of systems can make "rare" events common.
- Thanks to the Web, **malicious users** are everywhere.

But *some* projects can get away with ignoring some of these ideas.

Understand the needs of the project you are working on!

Course Wrap-Up
THE END