Introduction to Graphs
Graph Traversals

CS 311 Data Structures and Algorithms

Lecture Slides

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Review

Unit Overview
Tables & Priority Queues

Major Topics

- ✓ Introduction to Tables ← Lots of lousy implementations
- ✓ Priority Queues
- ✓ Binary Heap Algorithms
- ✓ Heaps & Priority Queues in the C++ STL
- ✓ 2-3 Trees
- ✓ Other self-balancing search trees
- ✓ Hash Tables
- ✓ Prefix Trees ← A special-purpose implementation: "the Radix Sort of Table implementations"
- ✓ Tables in the C++ STL & Elsewhere

Idea #1: Restricted Table (Priority Queues; Binary Heap Algorithms; Heaps & Priority Queues in the C++ STL)

Idea #2: Keep a tree balanced (2-3 Trees; Other self-balancing search trees)

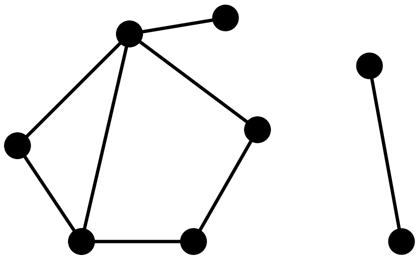
Idea #3: Magic functions (Hash Tables)

DONE

The Rest of the Course
Overview

Two Final Topics

- ✓ External Data
  - Previously, we dealt only with data stored in memory.
  - Suppose, instead, that we wish to deal with data stored on an external device, accessed via a relatively slow connection and available in sizable chunks (data on a disk, for example).
  - How does this affect the design of algorithms and data structures?
- Graph Algorithms
  - A **graph** models relationships between pairs of objects.
  - This is a very general notion. Algorithms for graphs often have very general applicability.

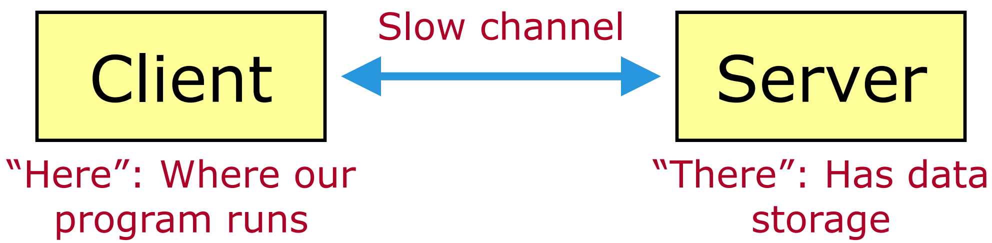


Drawing of a Graph

This usage of "graph" has nothing to do with the graph of a function. It is a different definition of the word.

Review
External Data [1/5]

We consider *data* that are accessed via a slow channel.



Typically, the channel transmits data in chunks: **blocks**.

Thus, *minimize the number of block accesses*.

External Sorting: Merge Sort variant

- Stable Merge works well with block-access data.
- Use temporary files for the necessary buffers.

External Table Implementation #1: Hash Table

- Open hashing, with each bucket stored in a small fixed number of blocks where possible, works well.

Review
External Data [2/5]

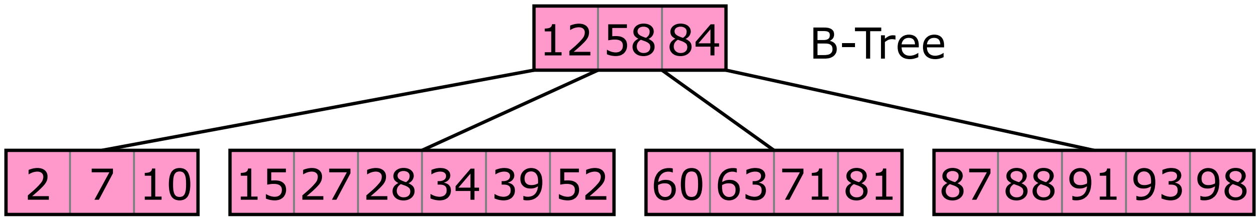
External Table Implementation #2: B-Tree

A **B-Tree of degree *m*** (*m* ≥ 3) is a ceiling(*m*/2) … *m* Tree.

- A node has ceiling(*m*/2)–1 … *m*–1 items.
- Except: The root can have 1 … *m*–1 items.
- All leaves are at the same level.
- Non-leaves have 1 more child than # of items.
- The order property holds, as for 2-3 Trees and 2-3-4 Trees.
- Degree = max # of children = # of items in an **over-full** node.

2-3 Tree = B-Tree of degree 3. 2-3-4 Tree = B-Tree of degree 4.

Shown is a B-Tree of degree 7.

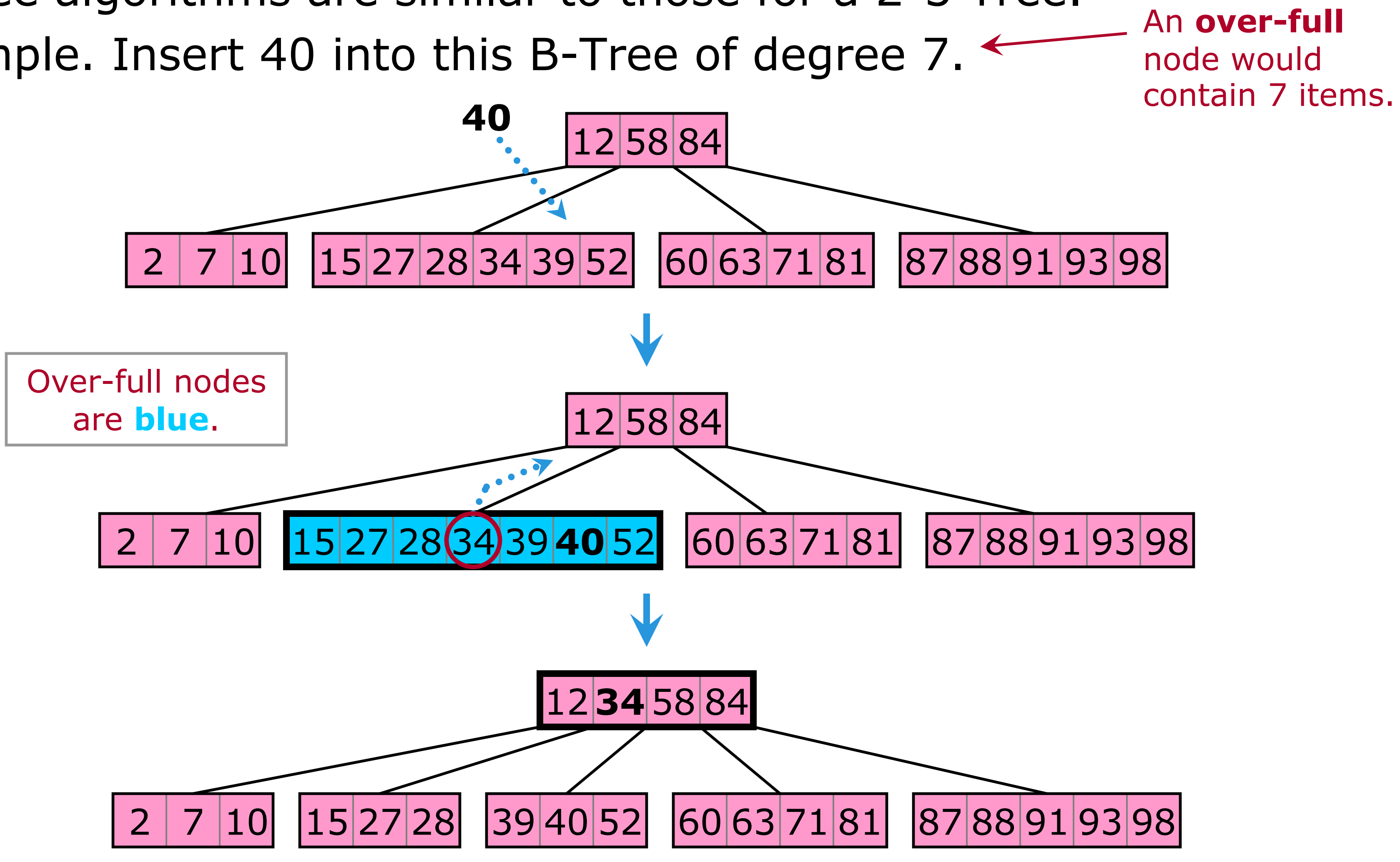


In practice, the degree may be much higher (for example, 50).

Review
External Data [3/5]

B-Tree algorithms are similar to those for a 2-3 Tree.

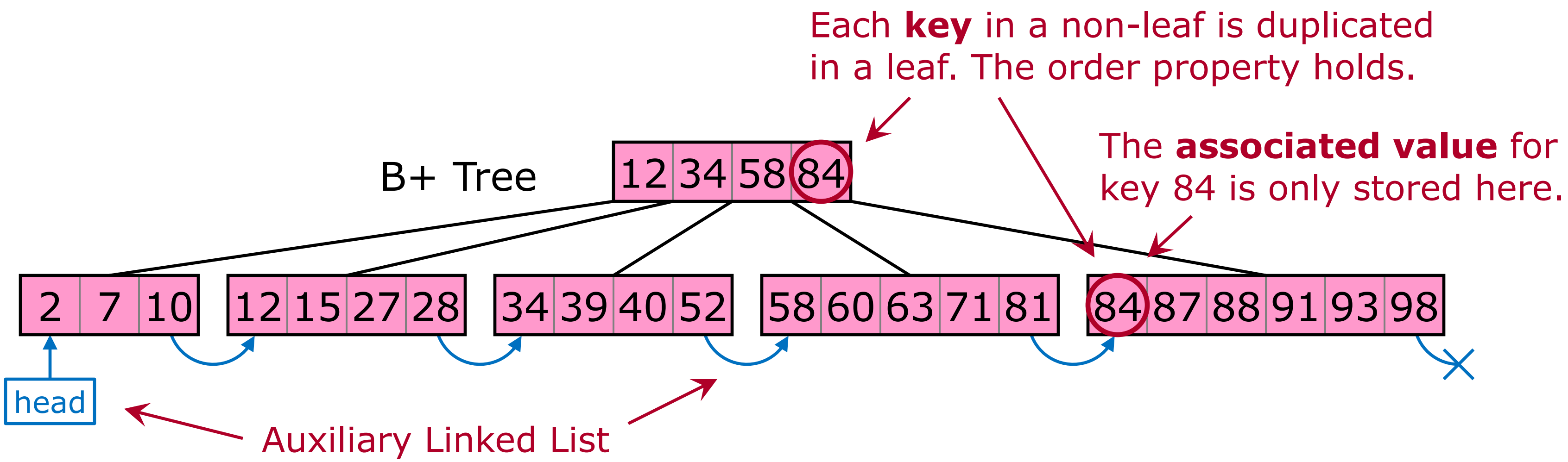
Example. Insert 40 into this B-Tree of degree 7.



Review
External Data [4/5]

There are a number of B-Tree variations. A common one: **B+ Tree**. This is the same as a B-Tree, except:

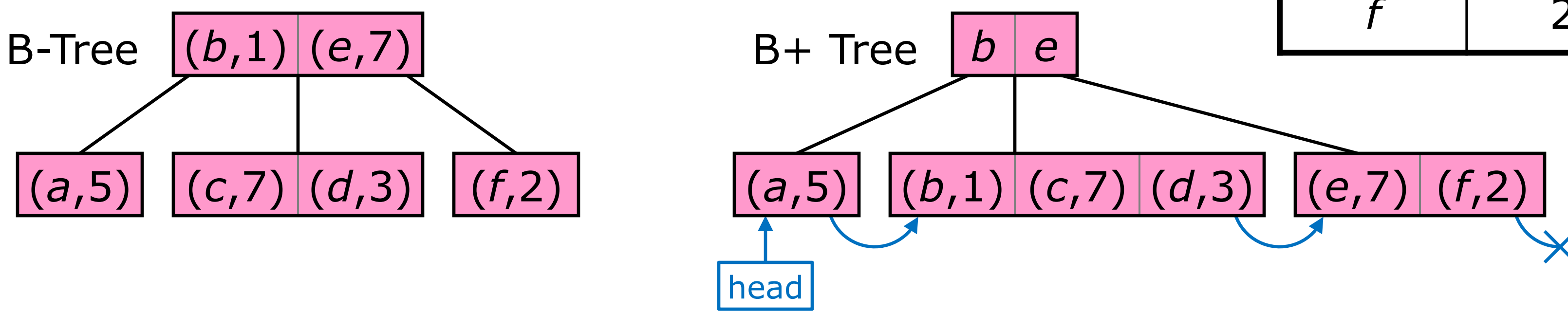
- Keys in non-leaf nodes are duplicated in the leaves, while maintaining the order property.
- Associated values are stored only in the leaves.
- Leaves are joined into an auxiliary Linked List. This minimizes the number of block accesses required for a traversal.



Review
External Data [5/5]

To the right is a Table dataset. Below on the left is a B-Tree holding this dataset. Below on the right is the corresponding B+ Tree. Both keys and associated values are shown.

Key	Value
a	5
b	1
c	7
d	3
e	7
f	2



Modern filesystems typically involve a B-Tree or variant internally.

- B+ Trees are a particularly common variant.

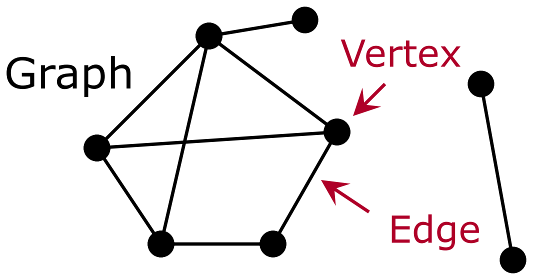
These trees are also used in relational-database implementation.

Introduction to Graphs

Introduction to Graphs
Definition

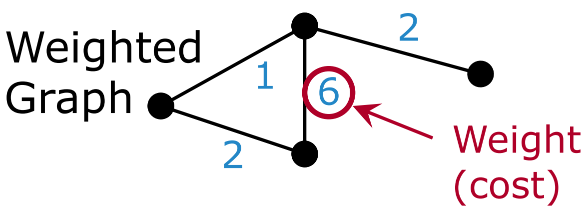
A **graph** consists of **vertices** and **edges**.

- An edge joins two vertices: its **endpoints**.
- 1 **vertex**, 2 vertices (Latin plural).
- Two vertices joined by an edge are **adjacent**; each is a **neighbor** of the other.



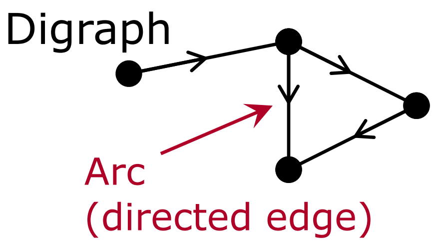
In a **weighted graph**, each edge has a **weight** (or **cost**).

- The weight is the resource expenditure required to use that edge.
- We typically choose edges to minimize the total weight of some kind of collection.



If we give each edge a direction, then we have a **directed graph**, or **digraph**.

- Directed edges are called **arcs**.



Introduction to Graphs
Applications

We use graphs to model:

- Networks
  - Vertices are nodes in network; edges are connections.
  - Examples
    - Communication
    - Transportation
    - Electrical
    - Worldwide Web (edges are links)
- State Spaces
  - Vertices are states; edges are transitions between states.
- Generally, situations in which objects are related in pairs:
  - Vertices are data-structure nodes; directed edges indicate pointers.
  - Vertices are people, edges indicate relationships (friendship?).
  - Vertices are tasks or events; edges join pairs that cannot occur at the same time (e.g., because of conflicting resource needs).

Introduction to Graphs
Representations [1/3]

How do we represent a graph in a computer program?

Adjacency matrix. 2-D array of 0/1 values.

- "Are vertices *i*, *j* adjacent?" in Θ(1) time.
- Finding all neighbors of a vertex is slow for large, sparse graphs.
  - **Sparse** graph: one with relatively few edges.

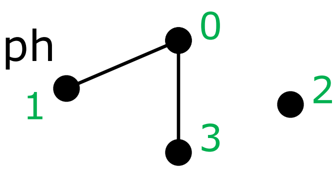
Adjacency lists. List of lists (arrays?). List *i* holds neighbors of vertex *i*.

- "Are vertices *i*, *j* adjacent?" in Θ(log *N*) time if lists are sorted arrays; Θ(*N*) if not.
- Finding all neighbors can be faster.

Both adjacency matrices and adjacency lists can be generalized to handle digraphs.

N: the number of vertices (more on this soon)

Graph



Adjacency Matrix

	0	1	2	3
0	0	1	0	1
1	1	0	0	0
2	0	0	0	0
3	1	0	0	0

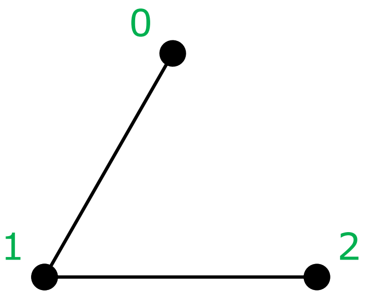
Labels are not part of the matrix.

Adjacency Lists

0: 1, 3
1: 0
2:
3: 0

Introduction to Graphs
Representations [2/3] (Try It!)

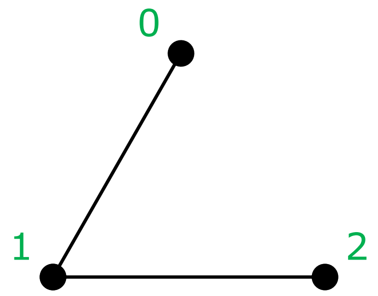
For the following graph, write (a) the adjacency matrix, and (b) adjacency lists.



Answers on the next slide.

Representations [3/3] (Try It!)

For the following graph, write (a) the adjacency matrix, and (b) adjacency lists.



Answers

(a)

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{array}$$

(b)

0: 1
1: 0, 2
2: 1

Green numbers are optional in the answer to part (a).

When an algorithm takes a graph, what is our "*n*"?

The number of vertices? The number of edges? Some combination?

We consider *both* the number of vertices and the number of edges.

- N = number of vertices
- M = number of edges

I use upper case (N, M) to make it clear that we are talking about vertices and edges, not the size of the input as a whole.

Adjacency matrices & adjacency lists are considered separately.

The *total* size of the input is:

- For an adjacency matrix: N^2. So $\Theta(N^2)$.
- For adjacency lists: $N + 2M$. So $\Theta(N + M)$.

The "2" is because each edge corresponds to two entries in the adjacency lists—one for each endpoint of the edge.

Some particular algorithm might have order (say) $\Theta(N + M \log N)$.

Graph Traversals

Graph Traversals
Introduction

We covered Binary Tree traversals: preorder, inorder, postorder.

We traverse graphs as well.

- Here, to **traverse** means to visit each vertex (once).
- Traditionally, graph traversal is viewed in terms of a “search”: visit each vertex searching for something.

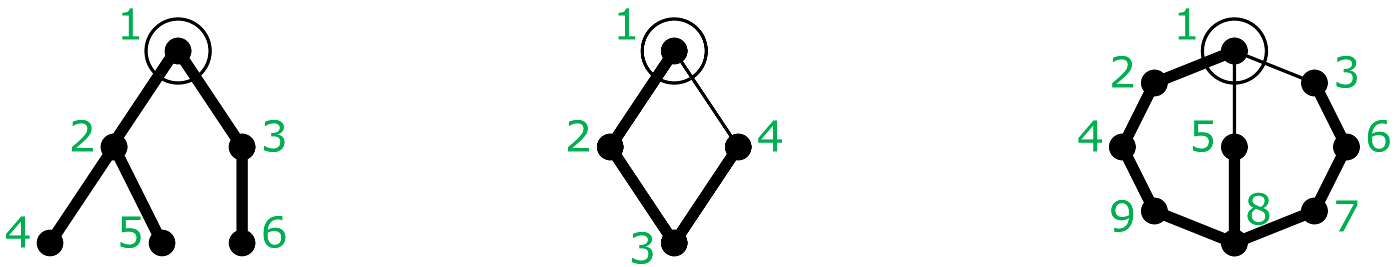
Two important graph traversals.

- **Depth-first search** (**DFS**)
  - Similar to a preorder Binary Tree traversal.
  - When we visit a vertex, give priority to visiting *its* unvisited neighbors (and when we visit one of them, we give priority to visiting *its* unvisited neighbors, etc.).
  - Result: we may proceed far from a vertex before visiting all its neighbors.
- **Breadth-first search** (**BFS**)
  - Visit all of a vertex’s unvisited neighbors before visiting their neighbors.
  - Result: Vertices are visited in order of distance from start.

Graph Traversals
DFS [1/2]

DFS has a natural recursive formulation:

- Given a *start* vertex, visit it, and mark it as visited.
- For each of the start vertex's neighbors:
  - If this neighbor is unvisited, then do a DFS with this neighbor as the start vertex.



DFS: 1, 2, 4, 5, 3, 6 DFS: 1, 2, 3, 4 DFS: 1, 2, 4, 9, 8, 5, 7, 6, 3

Recursion can be eliminated with a Stack. And we can be more intelligent than the brute-force method.

Graph Traversals
DFS [2/2]

Algorithm DFS

- Mark all vertices as unvisited.
- For each vertex:
  - Do algorithm DFS’ with this vertex as *start*.

Algorithm DFS’

- Set Stack to empty.
- Push *start* vertex on Stack.
- Repeat while Stack is non-empty:
  - Pop top of Stack.
  - If this vertex is not visited, then:
    - Visit it.
    - Push its not-visited neighbors on the Stack.

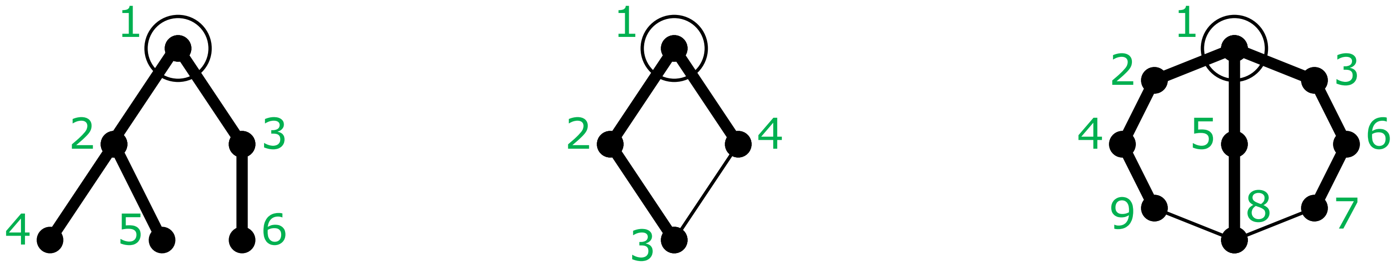
This part is all we need, if the graph is **connected** (all one piece). The above is only required for **disconnected** graphs.

TO DO

- Write a non-recursive function to do a DFS on a graph, given adjacency lists.

Done. See `graph_traverse.cpp`*.*

Graph Traversals
BFS [1/2]



BFS: 1, 2, 3, 4, 5, 6 BFS: 1, 2, 4, 3 BFS: 1, 2, 3, 5, 4, 6, 8, 9, 7

BFS is good for finding the shortest paths to other vertices.

Graph Traversals
BFS [2/2]

TO DO

- Modify our DFS function to do BFS.
  - BFS reverses the priority of neighbors vs. neighbors of neighbors.
  - Thus: replace the Stack with a Queue.

Done. See `graph_traverse.cpp`*.*

Graph Traversals
TO BE CONTINUED …

Graph Traversals will be continued next time.