Binary Heap Algorithms

CS 311 Data Structures and Algorithms

Lecture Slides

Wednesday, April 15, 2009

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Review
Binary Search Trees — Efficiency

	B.S.T. (balanced & average case)	Sorted Array	B.S.T. (worst case)
Retrieve	Logarithmic	Logarithmic	Linear
Insert	Logarithmic	Linear	Linear
Delete	Logarithmic	Linear	Linear

Binary Search Trees have poor worst-case performance.

But they have very good performance:

- On average.
- If balanced.
  - But we do not know an efficient way to make them *stay* balanced.

Can we efficiently keep a Binary Search Tree balanced?

Unit Overview
Tables & Priority Queues

Major Topics

- ✓ Introduction to Tables
- ✓ Priority Queues
- Binary Heap algorithms
- Heaps & Priority Queues in the C++ STL
- 2-3 Trees
- Other balanced search trees
- Hash Tables
- Prefix Trees
- Tables in various languages

Review
Introduction to Tables [1/4]

Our ultimate value-oriented ADT is **Table**. Three primary operations:

- Retrieve (by key).
- Insert (key-data pair).
- Delete (by key).

What do we use a Table for?

- To hold data accessed by key fields. For example:
  - Customers accessed by phone number.
  - Students accessed by student ID number.
  - Any other kind of data with an ID code.
- To hold “set” data.
  - Data in which the only question we ask is whether a key lies in the data set.
- To hold “arrays” whose indices are not nonnegative integers.
  - **`arr2["hello"] = 3;`**
- To hold array-like data sets that are **sparse**.
  - **`arr[6] = 1; arr[1000000000] = 2;`**

Review
Introduction to Tables [2/4]

How can we implement a Table? We know several lousy ways:

- A Sequence holding key-data pairs.
  - Sorted or unsorted.
  - Array-based or Linked-List-based.
- A Binary Search Tree holding key-data pairs.

	Sorted Array	Unsorted Array	Sorted Linked List	Unsorted Linked List	Binary Search Tree	Balanced** Binary Search Tree
Retrieve	Logarithmic	Linear	Linear	Linear	Linear	Logarithmic
Insert	Linear	Constant (?)*	Linear	Constant	Linear	Logarithmic
Delete	Linear	Linear	Linear	Linear	Linear	Logarithmic

*Amortized constant time, if we might have to reallocate. (Also, we are allowing multiple equivalent keys here. If we do not, then when we insert, we need to find any existing item with the given key, which is linear time.)

**Of course, we do not (yet!) know any way to guarantee the tree will *stay* balanced, unless we can restrict ourselves to read-only operations (no insert, delete).

Review
Introduction to Tables [3/4]

In special situations, the (amortized) constant-time insertion for an unsorted array and the logarithmic-time retrieve for a sorted array can be combined!

- Insert all data into an unsorted array, sort the array, then use Binary Search to retrieve data.
- This is a good way to handle Table data with **separate filling & searching phases** (and few or no deletes).
- Note: We will be talking about some complicated Table implementations. But *sometimes* a simple solution is the best.

Review
Introduction to Tables [4/4]

	Sorted Array	Unsorted Array	Sorted Linked List	Unsorted Linked List	Binary Search Tree	Balanced (how?) Binary Search Tree
Retrieve	Logarithmic	Linear	Linear	Linear	Linear	Logarithmic
Insert	Linear	Constant (?)	Linear	Constant	Linear	Logarithmic
Delete	Linear	Linear	Linear	Linear	Linear	Logarithmic

Idea #1: Restricted Table
- Perhaps we can do better if we do not implement a Table in its full generality.

Idea #2: Keep a Tree Balanced
- Balanced Binary Search Trees look good, but how to keep them balanced efficiently?

Idea #3: "Magic Functions"
- Use an unsorted array of key-data pairs. Allow array items to be marked as "empty".
- Have a "magic function" that tells the index of an item.
- Retrieve/insert/delete in constant time? (Actually no, but this is still a worthwhile idea.)

We will look at what results from these ideas:
- From idea #1: Priority Queues
- From idea #2: Balanced search trees (2-3 Trees, Red-Black Trees, B-Trees, etc.)
- From idea #3: Hash Tables

Review
Priority Queues — What a Priority Queue Is

A Priority Queue is a restricted-access Table.

- Just as a Queue is a restricted Sequence.
- A Priority Queue is not a Queue!

The key is called the “**priority**”.

We can:

- Retrieve (**getFront**) highest priority item.
- **Insert** any item (key, data).
- **Delete** highest priority item.

Review
Priority Queues — Applications, Implementation

A PQ is useful when we have items to process and some are more important than others.

PQs can be used to do sorting.

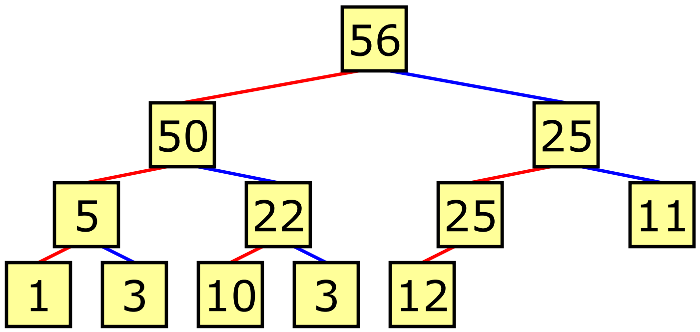
- Insert all items, then retrieve/delete all items. The resulting sequence is sorted by priority.
- Note: Once again, a sorted container gives us a sorting algorithm. However, as with Insertion Sort, instead of using a separate container to sort with, we prefer to use an in-place version of the algorithm. We will call this “Heap Sort”.

The most interesting thing about Priority Queues is their usual implementation: a Binary Heap.

Binary Heap Algorithms
What is a Binary Heap? — Definition

We define a **Binary Heap** (usually just **Heap**) to be a complete Binary Tree that

- Is empty,
- Or else
  - The root's key (priority) is ≥ than the key of each of the root's children, if any, and
  - Each of the root's subtrees is a Binary Heap.



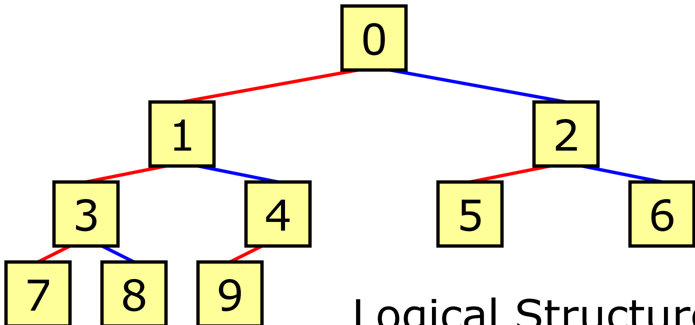
Notes

- This is a **Maxheap**. If we reverse the order, so that the root's key is ≤ than the keys of its children, we get a **Minheap**.
- The text presents Heap as an ADT with essentially the same operations as a Priority Queue. I am not doing this.
- As we will see, a Binary Heap is a good basis for an implementation of a Priority Queue.

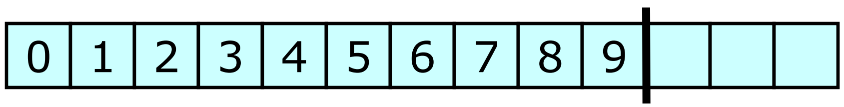
Binary Heap Algorithms
What is a Binary Heap? — Complete BTs (again)

We discussed an array implementation for a **complete** Binary Tree:

- Put the nodes in an array, in the order in which they would be added to a complete Binary Tree.
- No pointers/arrows/indices are required.
- We store *only* the **array** of data items and the **number** of nodes.



Logical Structure

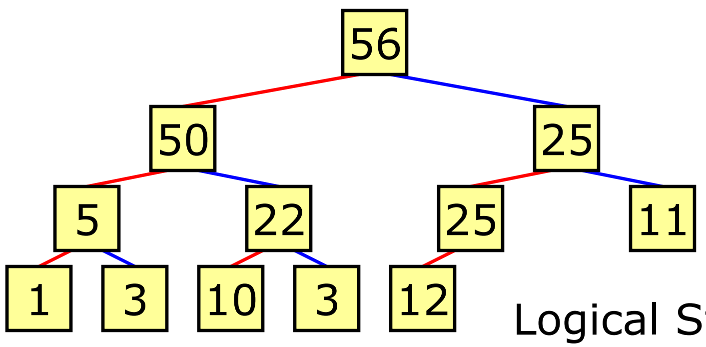


Physical Structure

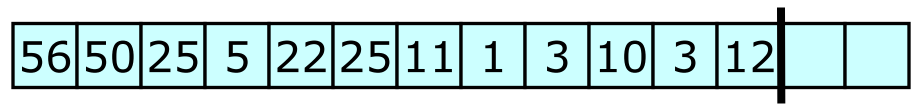
- Put the root, if any, at index 0.
- The left child of node k is at index $2k + 1$. It exists if $2k + 1 < size$.
- The right child is similar, but at $2k + 2$.
- The parent of node k is at index $(k - 1)/2$ [int division]. It exists if $k > 0$.

Binary Heap Algorithms
What is a Binary Heap? — Implementation

The usual implementation of a Binary Heap uses this array-based complete Binary Tree.



Logical Structure



Physical Structure

There are no required order relationships between siblings.

None of the standard traversals gives any sensible ordering.

In practice, we usually use "Heap" to mean a Binary Heap using this array representation.

In order to base a Priority Queue on a Heap, we need to know how to implement the operations.

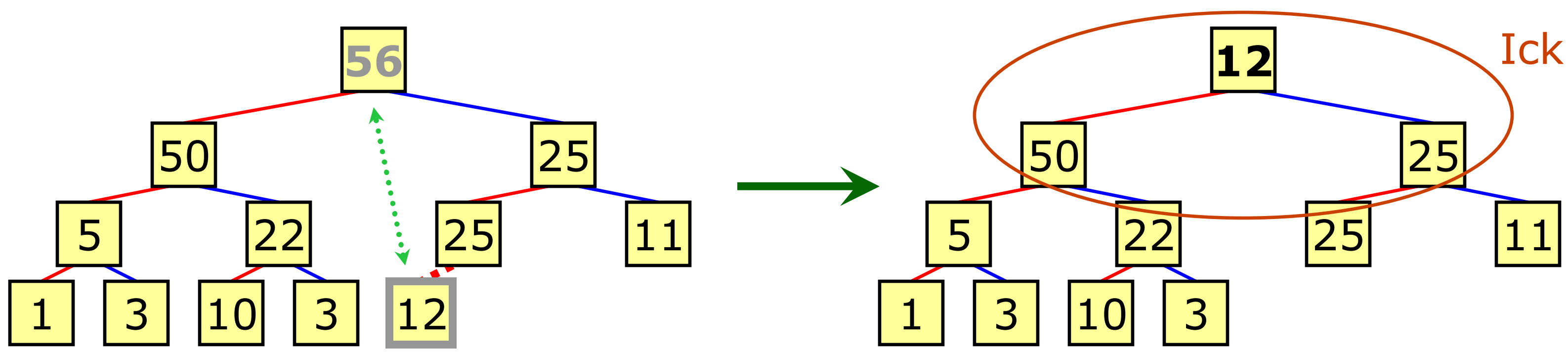
- getFront is easy (right?). Next we look at delete & insert.

Binary Heap Algorithms
Operations — Delete [1/2]

In a Priority Queue, we can delete the highest-priority item.

In a Maxheap, this corresponds to the root. How do we delete the **root item**, while maintaining the Heap properties?

- We cannot delete the **root node** (unless it is the only node).
- The Heap will have one less item, and so the **last node** must go away.
- But the **last item** is not going away.
- Solution: Move the last node’s item to the root; delete the last node.
  - We do this by swapping the items (which has other advantages, as we will see).

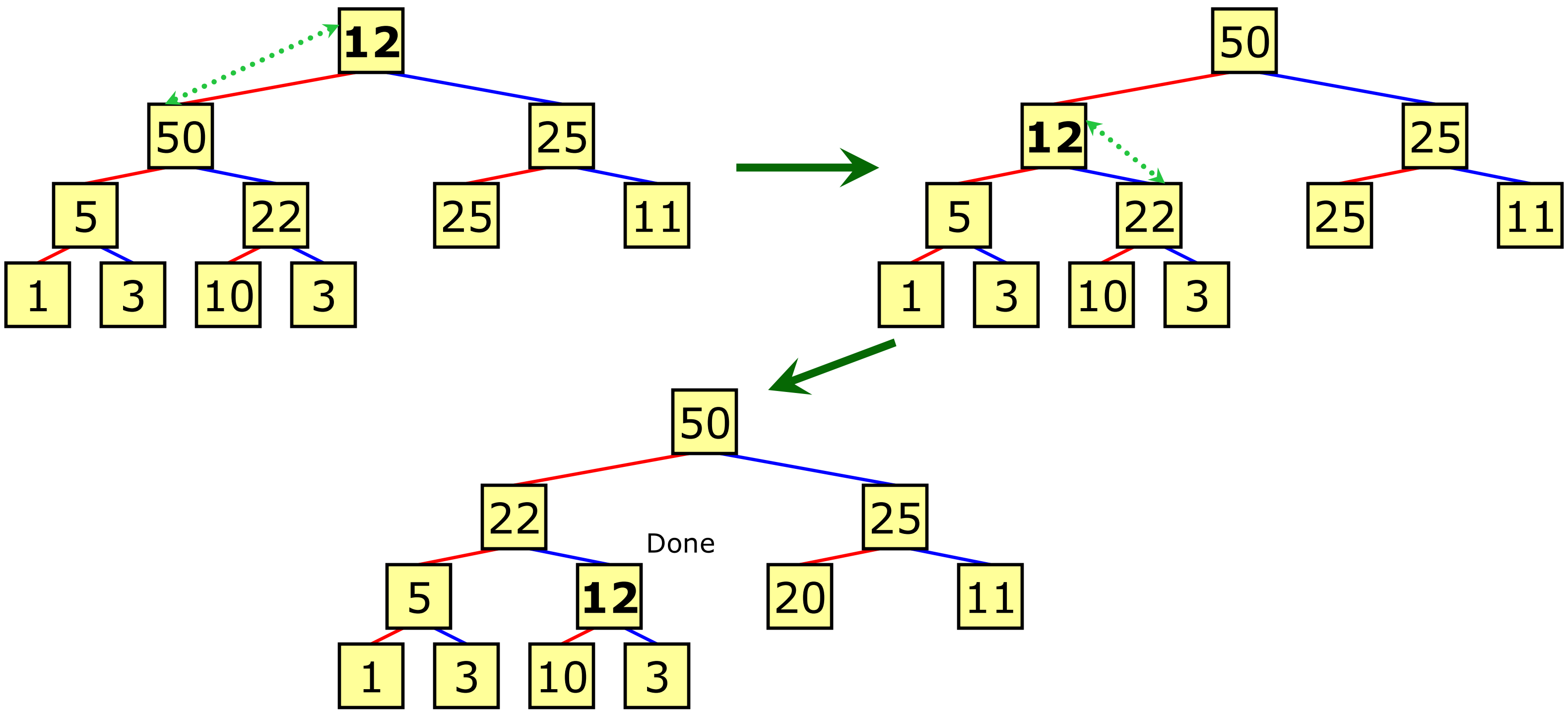


- But now we have another problem: This is no longer a Heap.
  - How do we fix it?

Binary Heap Algorithms
Operations — Delete [2/2]

To fix: “Trickle down” the new root item.

- If the root is ≥ all its children, stop.
- Otherwise, swap the root item with its **largest** child and recursively fix the proper subtree. (Why largest?)



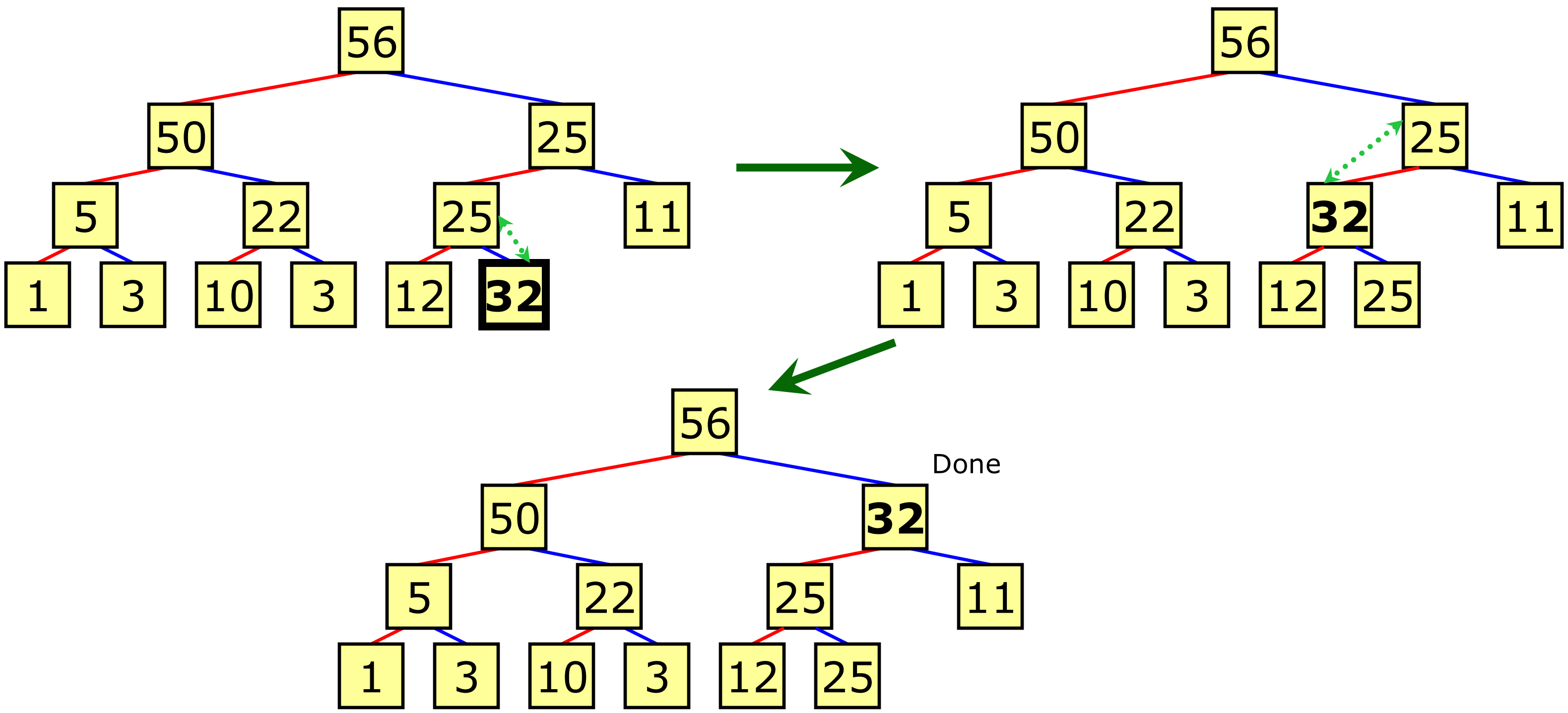
Binary Heap Algorithms
Operations — Insert

To insert into a Heap, add a new node at the end.

- But if we put our new value in this node, then we may not have a Heap.

Solution: “Trickle up”.

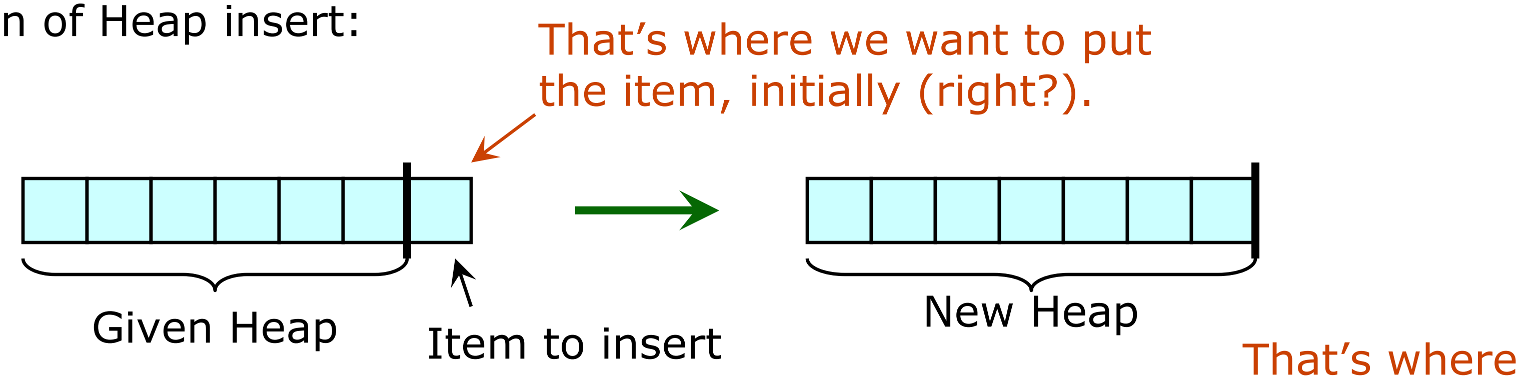
- If new value is greater than its parent, swap them. Repeat at new position.



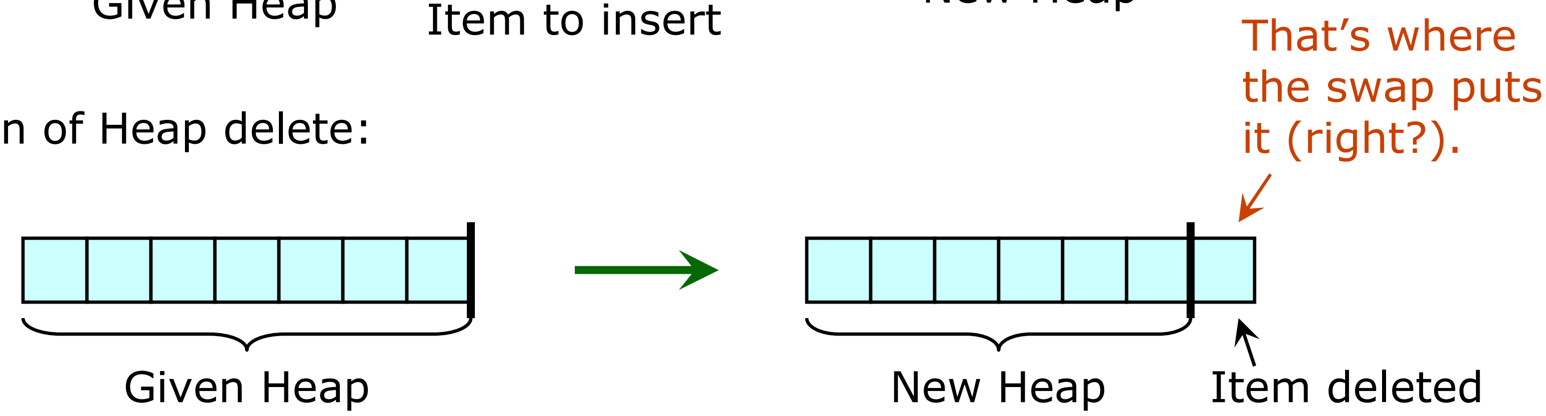
Binary Heap Algorithms
Operations — Using an Array

Heap insert and delete are usually given a random-access range. The item to insert or delete is last item of the range; the rest is a Heap.

- Action of Heap insert:



- Action of Heap delete:



Note that Heap algorithms can do **all** their work using **swap**.

- This usually allows for both speed and safety.

Binary Heap Algorithms
Efficiency

What is the order of the three main Priority Queue operations, if we use a Binary Heap implementation based on a complete Binary Tree stored in an array?

- **getFront**
  - Constant time.
- **insert**
  - Logarithmic time.
    - Assuming no reallocation, that is, assuming the array is large enough to hold the new item. As on the previous slide, the way that Heaps are **used** often guarantees that this is the case. (Linear time if possible reallocation.)
  - The number of operations is roughly the height of the tree. Since the tree is balanced, the height is $O(\log n)$.
- **delete**
  - Logarithmic time.

    Better than linear time!
  - No reallocation, of course. Other comments as for insert.

We conclude that a Heap is an excellent basis for an implementation of a Priority Queue.

Binary Heap Algorithms
Write It!

TO DO

- Write the Heap insert algorithm.
  - Prototype is shown below.
  - The item to be inserted is the final item in the given range.
  - All other items should form a Heap already.

Done. See **`heapalgs.h`**, *on the web page.*

```
// Requirements on types:
//     RAIter is a random-access iterator type.
template<typename RAIter>
void heapInsert(RAIter first, RAIter last);
```

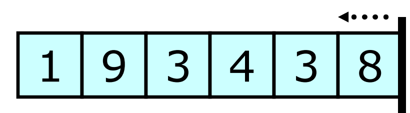
Binary Heap Algorithms
An Efficient "Make" Operation

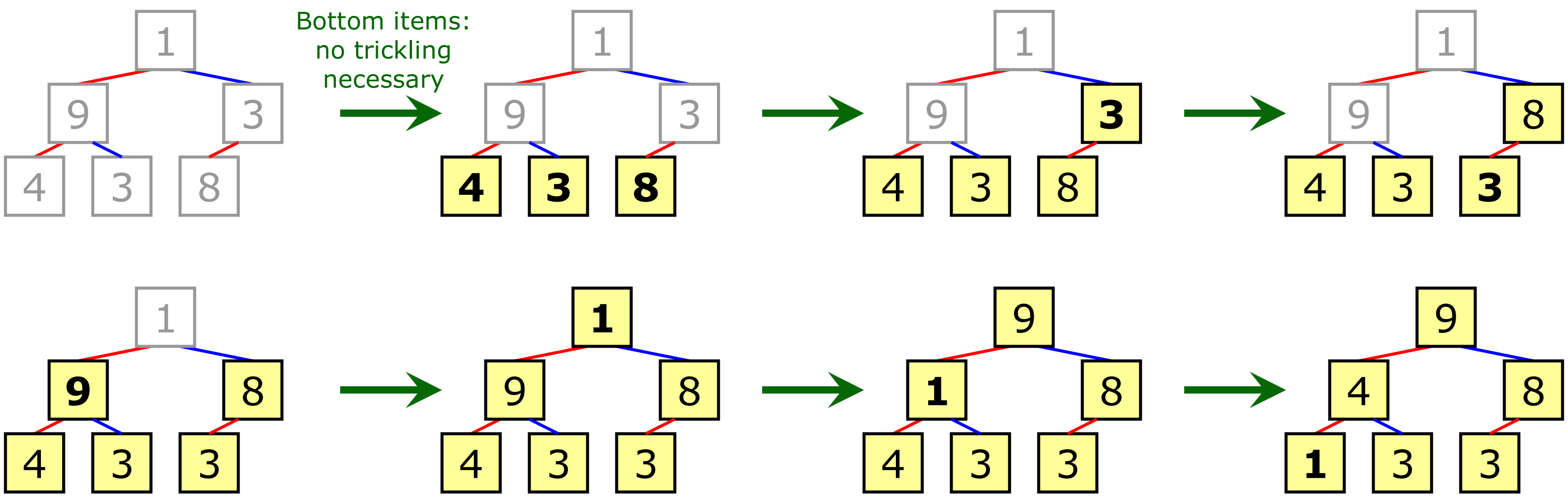
To turn a random-access range (array?) into a Heap, we *could* do n–1 Heap inserts.

- Each insert operation is $O(\log n)$, and so making a Heap in this way is $O(n \log n)$.

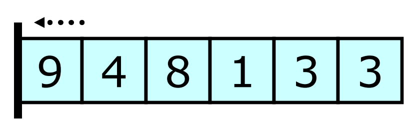
However, we can make a Heap **faster** than this.

- Place each item into a partially-made Heap, in **backwards order**.
- Trickle each item *down* through its descendants.
  - For most items, there are not very many of these.





This Heap "make" method is linear time!



Binary Heap Algorithms
Heap Sort — Introduction

Our last sorting algorithm is **Heap Sort**.

- This is a sort that uses Heap algorithms.
- We can think of it as using a Priority Queue, where the priority of an item is its value — except that the algorithm is in-place, using no separate data structure.
- Procedure: Make a Heap, then delete all items, using the delete procedure that places the deleted item in the top spot.
- We do a **make** operation, which is $O(n)$, and n getFront/delete operations, each of which is $O(\log n)$.
- Total: $O(n \log n)$.

Binary Heap Algorithms
Heap Sort — Properties

Heap Sort can be done in-place.

- We can create a Heap in a given array.
- As each item is removed from the Heap, put it in the array element that is removed from the Heap.
  - Starting the delete by swapping root and last items does this.
- Results
  - Ascending order, if we used a Maxheap.
  - Only constant additional memory required.
  - Reallocation is avoided.

Heap Sort uses less additional space than Introsort or array Merge Sort.

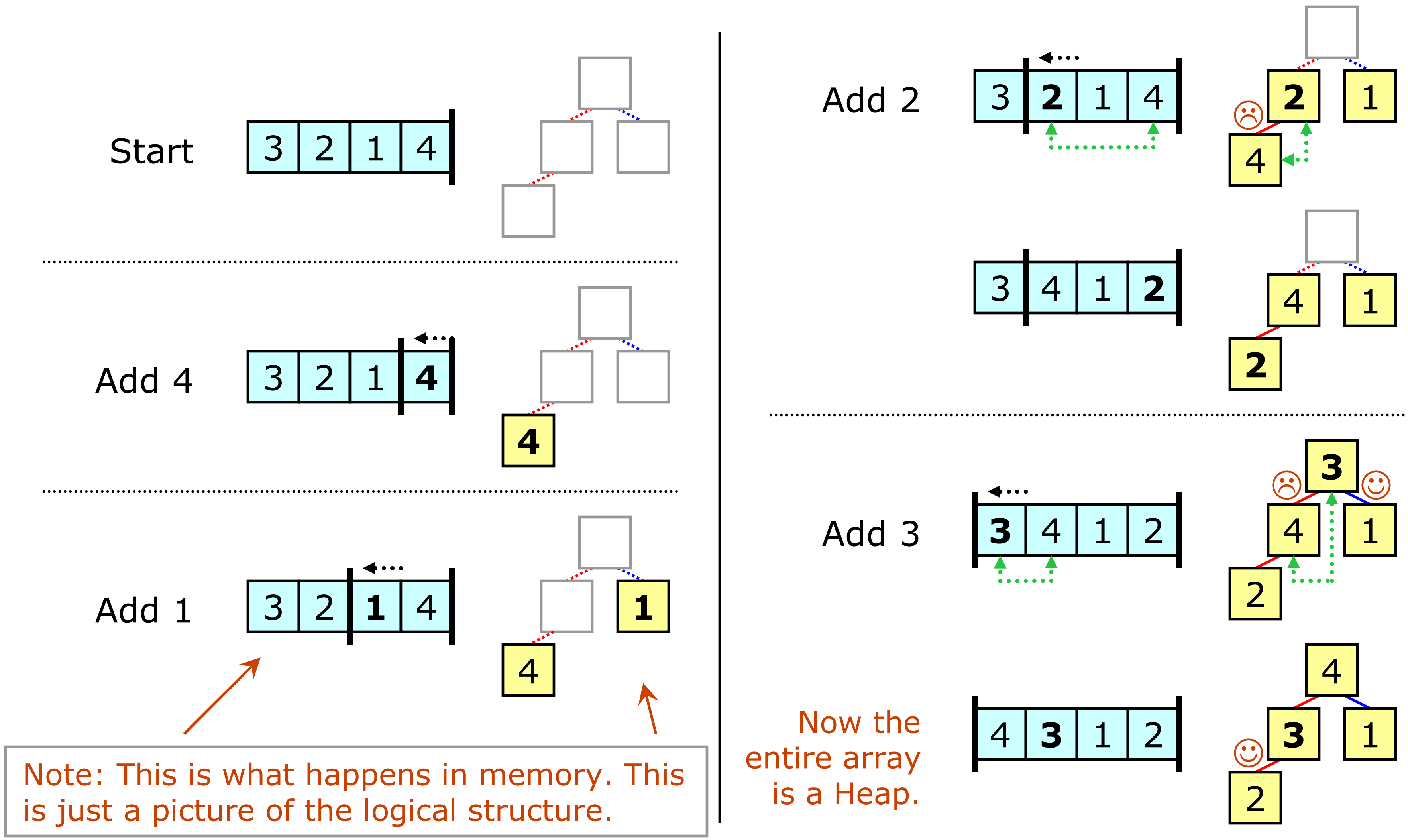
- Heap Sort: $O(1)$.
- Introsort: $O(\log n)$.
- Merge Sort on an array: $O(n)$.

Heap Sort also can easily be generalized.

- Doing Heap inserts in the middle of the sort.
- Stopping before the sort is completed.

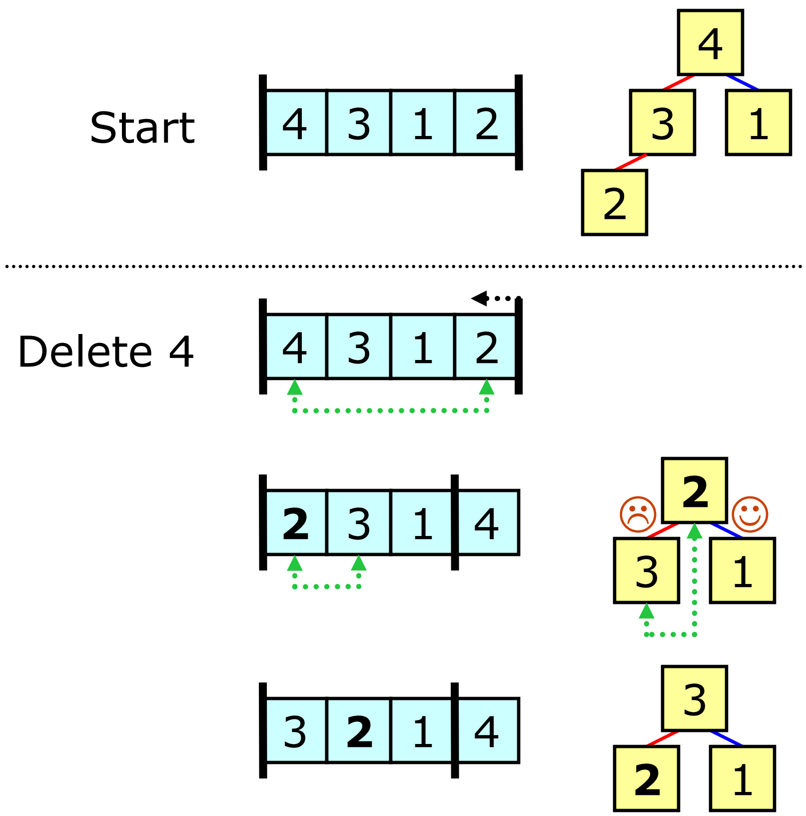
Binary Heap Algorithms
Heap Sort — Illustration [1/2]

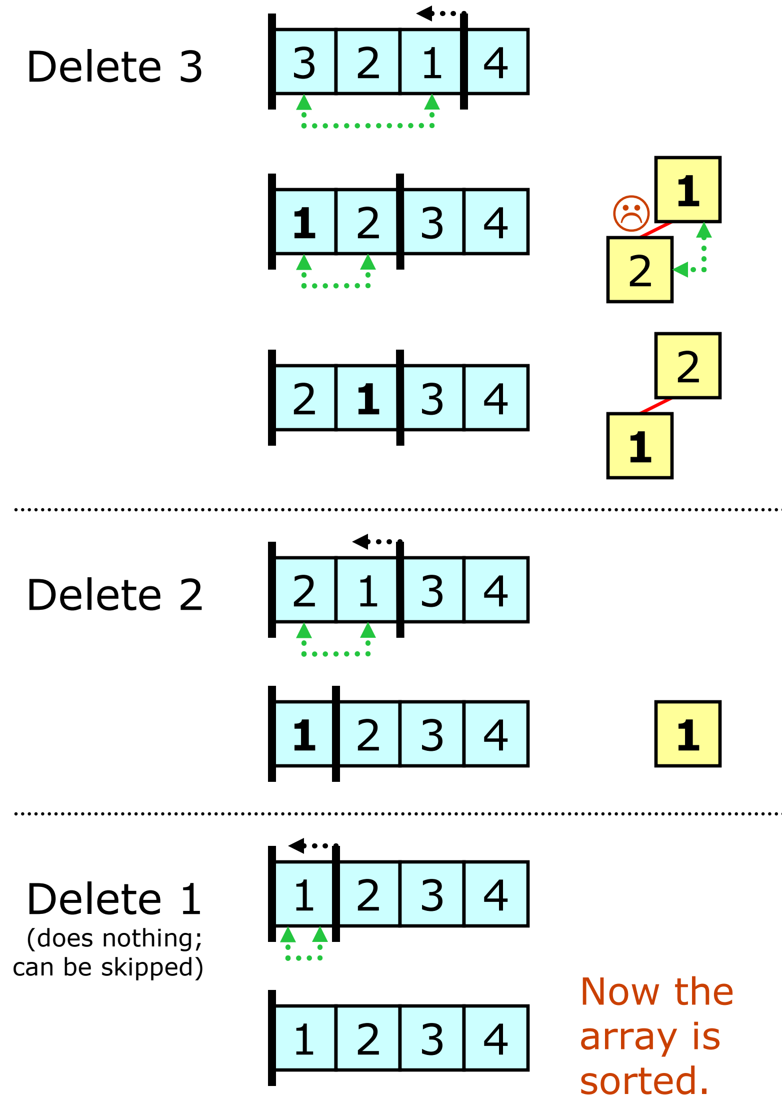
Below: Heap make operation. Next slide: Heap deletion phase.



Binary Heap Algorithms
Heap Sort — Illustration [2/2]

Heap deletion phase:





Binary Heap Algorithms
Heap Sort — Analysis

Efficiency ☺
- Heap Sort is $O(n \log n)$.

Requirements on Data ☹
- Heap Sort requires random-access data.

Space Usage ☺
- Heap Sort is in-place.

Stability ☹
- Heap Sort is not stable.

Performance on Nearly Sorted Data 😐
- Heap Sort is not significantly faster or slower for nearly sorted data.

We have seen these together before (Iterative Merge Sort on a Linked List), but never for an array.

Notes
- Heap Sort can be generalized to handle sequences that are modified (in certain ways) in the middle of sorting.
- Recall that Heap Sort is used by Introsort, when the recursion depth of Quicksort exceeds the maximum allowed.

Binary Heap Algorithms Thoughts

In practice, a Heap is not so much a data structure as it is an ordinary random-access sequence with a particular ordering property.

Associated with Heaps are a collection of algorithms that allow us to efficiently create Priority Queues and do comparison sorting.

- These **algorithms** are the things to remember.
- Thus the subject heading.