Radix Sort
Sorting in the C++ STL

CS 311 Data Structures and Algorithms
Lecture Slides
Monday, March 16, 2009

Glenn G. Chappell
Department of Computer Science
University of Alaska Fairbanks
CHAPPELLG@member.ams.org

© 2005–2009 Glenn G. Chappell
Unit Overview
Algorithmic Efficiency & Sorting

Major Topics
✓ • Introduction to Analysis of Algorithms
✓ • Introduction to Sorting
✓ • Comparison Sorts I
✓ • More on Big-O
✓ • The Limits of Sorting
✓ • Divide-and-Conquer
✓ • Comparison Sorts II
✓ • Comparison Sorts III
  • Radix Sort
  • Sorting in the C++ STL
Review
Introduction to Analysis of Algorithms

Efficiency
- General: using few resources (time, space, bandwidth, etc.).
- Specific: fast (time).

Analyzing Efficiency
- Determine how the size of the input affects running time, measured in steps, in the worst case.

Scalable
- Works well with large problems.

<table>
<thead>
<tr>
<th>Using Big-O</th>
<th>In Words</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>Constant time</td>
</tr>
<tr>
<td>$O(\log n)$</td>
<td>Logarithmic time</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>Linear time</td>
</tr>
<tr>
<td>$O(n \log n)$</td>
<td>Log-linear time</td>
</tr>
<tr>
<td>$O(n^2)$</td>
<td>Quadratic time</td>
</tr>
<tr>
<td>$O(b^n)$, for some $b &gt; 1$</td>
<td>Exponential time</td>
</tr>
</tbody>
</table>
Review
Introduction to Sorting — Basics, Analyzing

Sort: Place a list in order.

Key: The part of the data item used to sort.

Comparison sort: A sorting algorithm that gets its information by comparing items in pairs.

A general-purpose comparison sort places no restrictions on the size of the list or the values in it.

Five criteria for analyzing a general-purpose comparison sort:

- (Time) Efficiency
- Requirements on Data
- Space Efficiency
- Stability
- Performance on Nearly Sorted Data

In-place = no large additional storage space required (constant additional space).

1. All items close to proper places, OR
2. Few items out of order.
Review
Introduction to Sorting — Overview of Algorithms

There is no known sorting algorithm that has all the properties we would like one to have.

We will examine a number of sorting algorithms. Most of these fall into two categories: $O(n^2)$ and $O(n \log n)$.

- Quadratic-Time [$O(n^2)$] Algorithms
  - Bubble Sort
  - Selection Sort
  - Insertion Sort
  - Quicksort
  - Treesort (later in semester)

- Log-Linear-Time [$O(n \log n)$] Algorithms
  - Merge Sort
  - Heap Sort (mostly later in semester)
  - Introsort (not in the text)

- Special Purpose — Not Comparison Sorts
  - Pigeonhole Sort
  - Radix Sort
**Review**

**Comparison Sorts III — Quicksort: Description**

**Quicksort** is another divide-and-conquer algorithm. Procedure:

- Chooses a list item (the **pivot**).
- Do a **Partition**: put items less than the pivot before it, and items greater than the pivot after it.
- Recursively sort two sublists: items before pivot, items after pivot.

We did a simple pivot choice: the first item. Later, we improve this.

Fast Partition algorithms are in-place, but not stable.

- Note: In-place Partition does not give us an in-place Quicksort. Quicksort uses memory for recursion.
Review
Comparison Sorts III — Quicksort: Improvements

Unoptimized Quicksort is slow (quadratic time) on nearly sorted data and uses a lot of space (linear) for recursion.

Three Improvements

- Median-of-three pivot selection.
  - Improves performance on most nearly sorted data.
  - Requires random-access data.

- Tail-recursion elimination on larger recursive call.
  - Reduces space usage to logarithmic.

- Do not sort small sublists; finish with Insertion Sort.
  - General speed up.
  - May adversely affect cache hits.

With these optimizations, Quicksort is still $O(n^2)$ time.
Review
Comparison Sorts III — Quicksort: Analysis

Efficiency ☹
- Quicksort is $O(n^2)$.
- Quicksort has a very good $O(n \log n)$ average-case time. ☺☺

Requirements on Data ☹
- Non-trivial pivot-selection algorithms (median-of-three and others) are only efficient for random-access data.

Space Usage ☺
- Quicksort uses space for recursion.
  - Additional space: $O(\log n)$, if you are clever about it.
  - Even if all recursion is eliminated, $O(\log n)$ additional space is used.
  - This additional space need not hold any data items.

Stability ☹
- Efficient versions of Quicksort are not stable.

Performance on Nearly Sorted Data ☺
- A non-optimized Quicksort is slow on nearly sorted data: $O(n^2)$.
- Median-of-three Quicksort is $O(n \log n)$ on most nearly sorted data.
Review
Comparison Sorts III — Introsort: Description

In 1997, David Musser found out how to make Quicksort log-linear time.

- Keep track of the recursion depth.
- If this exceeds some bound (recommended: $2 \log_2 n$), then switch to Heap Sort for the current sublist.
  - Heap Sort is a general-purpose comparison sort that is log-linear time and in-place. We will discuss it in detail later in the semester.
- Musser calls this technique “introspection”. Thus, introspective sorting, or **Introsort**.
Review
Comparison Sorts III — Introsort: Diagram

Here is an illustration of how Introsort works.

- In practice, the recursion will be deeper than this.
- The Insertion-Sort call *might* not be done, due to its effect on cache hits.

Now, the list is nearly sorted. Finish with a (linear time!) Insertion Sort [??].

Tail recursion eliminated. (But it still counts toward the maximum recursion depth.)

When the sublist to sort is very small, do not recurse. Insertion Sort will finish the job later [??].

Recursion Depth Limit

When the recursion depth is too great, switch to Heap Sort to sort the current sublist.
Review
Comparison Sorts III — Introsort: Analysis

Efficiency ☺☺
- Introsort is $O(n \log n)$.
- Introsort also has an average-case time of $O(n \log n)$ [of course].
  - Its average-case time is just as good as Quicksort. ☺☺

Requirements on Data ☹
- Introsort requires random-access data.

Space Usage ☹
- Introsort uses space for recursion (or simulated recursion).
  - Additional space: $O(\log n)$ — even if all recursion is eliminated.
  - This additional space need not hold any data items.

Stability ☹
- Introsort is not stable.

Performance on Nearly Sorted Data ☹
- Introsort is not significantly faster or slower on nearly sorted data.
Radix Sort
Background

We have looked in detail at six general-purpose comparison sorts. Now we look at two sorting algorithms that do not use a comparison function:

- Pigeonhole Sort.
- Radix Sort.

Later in the semester, we will look closer at Heap Sort, which is a general-purpose comparison sort, but which can also be conveniently modified to handle other situations.
Radix Sort
Preliminaries: Pigeonhole Sort — Description

Suppose we have a list to sort, and:
- Keys lie in a **small fixed set of values**.
- Keys can be used to **index an array**.
  - E.g., they might be small-ish nonnegative integers.

Procedure
- Make an array of empty lists (**buckets**), one for each possible key.
- Iterate through the given list; insert each item at the end of the bucket corresponding to its value.
- Copy items in each bucket, in order, back to the original list.

Time efficiency: **linear time**, if written properly.
- How is this possible? Answer: We are not doing general-purpose comparison sorting. Our $\Omega(n \log n)$ bound does not apply.

This algorithm is often called **Pigeonhole Sort**.
- It is not very practical; it requires a very limited set of keys.
- Pigeonhole Sort is stable, and uses linear additional space.
Radix Sort
Preliminaries: Pigeonhole Sort — Write It

TO DO

- Write a function to do Pigeonhole Sort.
Radix Sort
Description

Using Pigeonhole Sort, we can design a practical algorithm: Radix Sort. Suppose we want to sort a list of strings (in some sense):

- Character strings.
- Numbers, considered as strings of digits.
- Short-ish sequences of some other kind.

Call the entries in a string “characters”.
- These need to be valid keys for Pigeonhole Sort.
- In particular, we must be able to use them as array indices.

The algorithm will arrange the list in lexicographic order.
- This means sort first by first character, then by second, etc.
- For strings of letters, this is alphabetical order.
- For positive integers (padded with leading zeroes), this is numerical order.

Radix Sort Procedure
- Pigeonhole Sort the list using the last character as the key.
- Take the list resulting from the previous step and Pigeonhole Sort it, using the next-to-last character as the key.
- Continue ...
- After re-sorting by first character, the list is sorted.
Radix Sort Example

Here is the list to be sorted.

- 583 508 183 90 223 236 924 4 426 106 624

We first sort them by the units digit, using Pigeonhole Sort.

- 90 583 183 223 924 4 624 236 426 106 508

Then Pigeonhole Sort again, based on the tens digit, in a stable manner (note that the tens digit of 4 is 0).

- 4 106 508 223 924 624 426 236 583 183 90

Again, using the hundreds digit.

- 4 90 106 183 223 236 426 508 583 624 924

And now the list is sorted.
Radix Sort

Write It, Comments

TO DO

- Write Radix Sort for small-ish positive integers.

Comments

- Radix Sort makes very strong assumptions about the values in the list to be sorted.
- It requires linear additional space.
- It is stable.
- It does not perform especially well or badly on nearly sorted data.
- Of course, what we really care about is speed. See the next slide.
Radix Sort
Efficiency [1/2]

How Fast is Radix Sort?

- Fix the number of characters and the character set.
- Then each sorting pass can be done in linear time.
  - Pigeonhole Sort with one bucket for each possible character.
- And there are a fixed number of passes.
- Thus, Radix Sort is $O(n)$: linear time.

How is this possible?

- Radix Sort is a sorting algorithm. However, again, it is neither general-purpose nor a comparison sort.
  - It places restrictions on the values to be sorted: not general-purpose.
  - It gets information about values in ways other than making a comparison: not a comparison sort.
- Thus, our argument showing that $\Omega(n \log n)$ comparisons were required in the worst case, does not apply.
Radix Sort
Efficiency [2/2]

In practice, Radix Sort is not really as fast as it might seem.

- There is a hidden logarithm. The number of passes required is equal to the length of a string, which is something like the logarithm of the number of possible values.
- If we consider Radix Sort applied to a list in which all the values might be different, then it is in the same efficiency class as normal sorting algorithms.

However, in certain special cases (e.g., big lists of small numbers) Radix Sort can be a useful technique.
Sorting in the C++ STL
Specifying the Interface

Iterator-based sorting functions can be specified two ways:
- Given a range
  - “last” is actually just past the end, as usual.

```
template<typename Iterator>
void sortIt(Iterator first, Iterator last);
```

- Given a range and a comparison.

```
template<typename Iterator, typename Ordering>
void sortIt(Iterator first, Iterator last, Ordering compare);
```

“`compare`”, above, should be something you can use to compare two values.
- “`compare(val1, val2)`” should be a legal expression, and should return a `bool`: true if `val1` comes before `val2` (think “less-than”).
- So `compare` can be a function (passed as a function pointer).
- It can also be an object with `operator()` defined: a `function object`. 
The C++ Standard Template Library has six sorting algorithms:

- Global function `std::sort`
- Global function `std::stable_sort`
- Member function `std::list<T>::sort`
- Global functions `std::partial_sort` and `partial_sort_copy`.
- Combination of two global functions: `std::make_heap` & `std::sort_heap`

We now look briefly at each of these.
Sorting in the C++ STL
Overview of the Algorithms [2/4]

Function **std::sort**, in `<algorithm>`
- Global function.
- Takes two random-access iterators and an optional comparison.
- $O(n^2)$, but has $O(n \log n)$ average-case. Not stable.
  - This should become $O(n \log n)$ in the forthcoming revised C++ standard.
  - It is currently $O(n \log n)$ in good STL implementations.
- Algorithm used:
  - Quicksort is what the standards committee was thinking.
  - Introsort is what good implementations now use.
  - Other algorithms (Heap Sort?) are possible, but unlikely.

Function **std::stable_sort**, in `<algorithm>`
- Global function.
- Takes two random-access iterators and an optional comparison.
- $O(n \log n)$. Stable.
- Algorithm used: probably Merge Sort, general sequence version.
Function `std::list<T>::sort`, in `<list>`

- Member function. Sorts only objects of type `std::list<T>`.
- Takes either no parameters or a comparison.
- $O(n \log n)$. Stable.
- Algorithm used: probably Merge Sort, Linked-List version.
We will look at the last two STL algorithms in more detail later in the semester, when we cover Priority Queues and Heaps:

- Functions `std::partial_sort` and `std::partial_sort_copy`, in `<algorithm>`
  - Global functions.
  - Take three random-access iterators and an optional comparison.
  - $O(n \log n)$. Not stable.
  - Solve a more general problem than comparison sorting.
  - Algorithm used: probably Heap Sort.

- Combination: `std::make_heap & std::sort_heap`, in `<algorithm>`
  - Both Global functions.
  - Both take two random-access iterators and an optional comparison.
  - Combination is $O(n \log n)$. Not stable.
  - Solves a more general problem than comparison sorting.
  - Algorithm used: almost certainly Heap Sort.
Algorithm `std::sort` is declared in the header `<algorithm>`. Call it with two iterators:

```cpp
vector<int> v;
std::sort(v.begin(), v.end());
// Ascending order
```

Or use two iterators and a comparison:

```cpp
std::sort(v.begin(), v.end(), std::greater<int>());
// Descending order
```

- Class template `std::greater` is defined in `<functional>`. Use `std::stable_sort` similarly to `std::sort`. 

Default constructor call. We can only pass an `object`, not a `type`. 

16 Mar 2009  
CS 311 Spring 2009
When sorting a `std::list`, use the `sort` member function:

```cpp
#include <list>

std::list<int> myList;

myList.sort();  // Ascending order
myList.sort(std::greater<int>());  // Descending order
```