Hash Tables
Prefix Trees

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Prefix Trees

CS 311 Data Structures and Algorithms
Lecture Slides
Monday, November 30, 2009

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Review
Where Are We? — The Big Problem

Our problem for much of the rest of the semester:

- Store: a collection of data items, all of the same type.
- Operations:
  - Access items [one item: retrieve/find, all items: traverse].
  - Add new item [insert].
  - Eliminate existing item [delete].
- All this needs to be efficient in both time and space.

A solution to this problem is a **container**.

**Generic containers**: those in which client code can specify the type of data stored.
Unit Overview
Tables & Priority Queues

Major Topics
- Introduction to Tables
- Priority Queues
- Binary Heap algorithms
- Heaps & Priority Queues in the C++ STL
- 2-3 Trees
- Other balanced search trees
- Hash Tables
- Prefix Trees
- Tables in various languages

Lots of lousy implementations

Idea #1: Restricted Table
Idea #2: Keep a Tree Balanced
Idea #3: “Magic Functions”

- (part) -

30 Nov 2009  CS 311 Fall 2009
Review
Introduction to Tables

<table>
<thead>
<tr>
<th>Idea #1: Restricted Table</th>
<th>Idea #2: Keep a Tree Balanced</th>
<th>Idea #3: “Magic Functions”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perhaps we can do better if we do not implement a Table in its full generality.</td>
<td>Balanced Binary Search Trees look good, but how to keep them balanced efficiently?</td>
<td>Use an unsorted array of key-data pairs. Allow array items to be marked as “empty”.</td>
</tr>
<tr>
<td>Have a “magic function” that tells the index of an item.</td>
<td>Retrieve/insert/delete in constant time? (Actually no, but this is still a worthwhile idea.)</td>
<td></td>
</tr>
</tbody>
</table>

We will look at what results from these ideas:
- From idea #1: Priority Queues
- From idea #2: Balanced search trees (2-3 Trees, Red-Black Trees, B-Trees, etc.)
- From idea #3: Hash Tables

<table>
<thead>
<tr>
<th>Sorted Array</th>
<th>Unsorted Array</th>
<th>Sorted Linked List</th>
<th>Unsorted Linked List</th>
<th>Binary Search Tree</th>
<th>Balanced (how?) Binary Search Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retrieve</td>
<td>Logarithmic</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>Insert</td>
<td>Linear</td>
<td>Constant??</td>
<td>Linear</td>
<td>Constant</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>Delete</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Logarithmic</td>
</tr>
</tbody>
</table>
Overview of Advanced Table Implementations

We will cover the following advanced Table implementations.

- **Balanced Search Trees**
  - Binary Search Trees are hard to keep balanced, so to make things easier we allow more than 2 children:
    - **2-3 Tree**
      - Up to 3 children
    - **2-3-4 Tree**
      - Up to 4 children
    - **Red-Black Tree**
      - Binary-tree representation of a 2-3-4 tree
  - Or back up and try a balanced Binary Tree again:
    - **AVL Tree**

- Alternatively, forget about trees entirely:
  - **Hash Tables**
  - Finally, “the Radix Sort of Table implementations”:
    - **Prefix Tree**
Review
2-3 Trees [1/4]

A Binary-Search-Tree style node is a **2-node**.

- This is a node with 2 subtrees and 1 data item.
- The item’s value lies between the values in the two subtrees.

In a “2-3 Tree” we also allow a node to be a **3-node**.

- This is a node with 3 subtrees and 2 data items.
- Each of the 2 data items has a value that lies between the values in the corresponding pair of consecutive subtrees.

Later, we will look at “2-3-4 trees”, which can also have **4-nodes**.
A **2-3 Search Tree** (generally we just say **2-3 Tree**) is a tree with the following properties.

- All nodes contain either 1 or 2 data items.
  - If 2 data items, then the first is $\leq$ the second.
- All leaves are at the same level.
- All non-leaves are either 2-nodes or 3-nodes.
  - They must have the associated order properties.

To **retrieve** in a 2-3 Tree:

- Begin at the root, and go down, using the order properties, until the item is found, or clearly not in the tree.

To **traverse** a 2-3 Tree:

- Use the appropriate generalization of inorder traversal.
- Items are visited in sorted order.
To **insert** in a 2-3 Tree:

- Find the leaf that the new item should go in.
- If it fits, then simply put it in.

- Otherwise, there is an overfull node. Split it, and move the middle item up, either recursively inserting it in the parent, or else creating a new root.
Review
2-3 Trees [4/4]

To **delete** in a 2-3 Tree:
- Find the item. If it is not in a leaf, swap with its successor.
- Do the recursive delete-a-leaf procedure.

To delete-a-leaf:
- **Easy Case**: If the item is in a node with another item, simply remove it.
- **Semi-Easy Case**: Otherwise, if the node has a consecutive sibling with two items, do a rotation with the parent.
- **Hard Case**: Otherwise, bring the parent down, combining it with a consecutive sibling.
  - Use recursive delete-a-leaf on the parent.

When doing a recursive “delete-a-leaf” on a non-leaf node, drag along subtrees.
In a **2-3-4 Tree**, we also allow 4-nodes.

![2-3-4 Tree Diagram]

The insert and delete algorithms are not terribly different from those of a 2-3 Tree.
- They are a little more complex.
- And they tend to be a little faster.
A very efficient kind of balanced search tree is a **Red-Black Tree**.

- This is a Binary-Search Tree representation of a 2-3-4 tree.
- Each node in a Red-Black Tree is either **red** or **black**.
- Each node in the 2-3-4 Tree corresponds to a **black node**.
- The **red nodes** are the extra ones we need to add.
- Red-Black Trees may not be balanced (in the strict sense). However, each path from the root to a leaf must pass through the same number of **black nodes**.

[Diagram of a 2-3-4 tree and its corresponding Red-Black Tree]
Review
Other Balanced Search Trees [3/3]

All balanced search trees (2-3 Trees, 2-3-4 Trees, AVL Trees, etc.) have:

- $O(\log n)$ retrieve, insert, delete.
- $O(n)$ traverse (sorted).

Retrieve & Sorted Traverse

- For Red-Black Trees and AVL Trees, use the B.S.T. algorithms (traverse = inorder traverse).
- For 2-3 Trees and 2-3-4 Trees, use the obvious generalization of the B.S.T. algorithms.

Insert & Delete

- These are more complicated.
- For 2-3 Trees, we looked at the algorithms in some detail.
- The 2-3-4 Trees and Red-Black Trees, the algorithms use the same ideas.

Best overall performance for in-memory data, when we mix up retrieves, inserts, and deletes.
A **Hash Table** is a Table implementation that uses a **hash function** for key-based look-up.

- A Hash Table is generally implemented as an array. The index used is the output of the hash function.

**Needed:**
- **Hash function.**
- **Collision resolution** method.
  - **Collision**: hash function gives same output for different keys.
Review
Hash Tables — Good Hash Functions

A hash function **must**:  
- Take a valid key and return an integer.  
- Be **deterministic**.  
  - Its value depends only on its input (the key). Using the same input multiple times results in the same output each time.

A **good** hash function:
- Can be computed quickly.  
- Spreads out its results evenly over the possible output values.  
  - To help spread out the results, some implementations give the Hash Table a **prime** number of locations.  
- Turns patterns in its input into random-looking output.

Each key type has its own hash function.  
- For client-defined key types, a hash function must be provided by the client.  
- Can put different key types, each with its own hash function, in the same Hash Table.  
- Hash Table sends the output of the provided hash function through a secondary function ("%"?) to make the output a valid index.
Collision Resolution Methods — Type 1: **Open Addressing**

- Hash Table is an array. Each location holds one key-data pair, "empty", or "deleted".
- Search in a sequence of locations (the **probe sequence**), beginning at the location given by the hashed key.
- **Linear probing**: \( t, t+1, t+2, \text{ etc.} \)
  - Tends to form **clusters**.
- **Quadratic probing**: \( t, t+1^2, t+2^2, \text{ etc.} \)
- **Double hashing**: Use another hash function to help determine the probe sequence.
Collision Resolution Methods — Type 2: “Buckets”

- Hash Table is an array of data structures, each of which can hold multiple key-data pairs.
- Array locations are **buckets**.
- **Separate chaining**: Each bucket is a Linked List.
  - This is very common.
Sometimes it is necessary to remake the Hash Table.

- All implementations have performance degradation as the number of data items rises.

In these cases, we need to do a reallocate-and-copy, as we did with smart arrays.

This is one of the downsides of Hash Tables.
A perfect hash function (one without collisions) results in insert, delete, and retrieve operations that are $O(1)$.

- In practice, we cannot guarantee this, *if* we allow insert & delete operations.
- But this might be a good idea, for a fixed data set (no insert/delete).

In the **worst case**, all items get the same hashed value, and so collisions happen nearly all the time.

- Thus, retrieve is linear time (worst case), for most implementations.
- *But what if our buckets are Red-Black Trees?*

However, we generally use a Hash Table when we are interested in **average-case** performance.

The average-case performance of a Hash Table can be analyzed based on the **load factor**.

- The *load factor*, denoted by $\alpha$, is:
  
  \[
  \frac{\text{(# of items present)}}{\text{(# of locations in table)}}
  \]

- We generally want $\alpha$ to be small. In the following slides, we will assume $\alpha$ is significantly less than 1 (less than $2/3$, maybe?).

- We will also assume, *for now*, that no Table-remake is required.
For example, consider separate chaining.

- **Worst Case**
  - Insert is constant time, assuming we do not search.
    - We can avoid a search, if we allow duplicate keys.
  - Retrieve and delete require a search: linear time.
  - Similarly, if we do not allow duplicate keys, then insert requires a search, and so is linear time.

- **Average Case**
  - The average number of items in a bucket is $\alpha$ (the load factor).
  - Thus, the average number of comparisons required for a search resulting in NOT FOUND is $\alpha$.
  - The average number of comparisons required for a search resulting in FOUND is approximately $1 + \alpha/2$.
  - This applies to operations requiring a search: retrieve and delete certainly, insert maybe. Insert without search is constant time.
Hash Tables
Efficiency — Open Addressing

With open addressing, retrieve, insert, and delete all require a search, even if duplicate keys are allowed.

Worst Case

- Linear time.

Average Case

- For linear probing:
  - NOT FOUND: \((1/2)[1+1/(1-\alpha)]^2\).
  - FOUND: \((1/2)[1+1/(1-\alpha)]\).

- For quadratic probing:
  - NOT FOUND: \(1/(1-\alpha)\).
  - FOUND: \(-\ln(1-\alpha)/\alpha\).

- Again:
  - We assume \(\alpha\) is significantly less than 1, and that the Table-remake operation is not done.
  - The efficiency of insert, delete, and retrieve is essentially the same in all cases.
Hash Tables
Efficiency — Traverse

Hash Table traverse can be slow, because of the empty locations.  
- Assume:
  - Either open addressing is used, or else buckets are implemented using structures that can be traversed in linear time.
  - We do not want a sorted traverse.
- Then traverse is $O(n + b)$, where $n$ is the number of items in the Hash Table, and $b$ is the number of locations (buckets?).

A speed-up: Use an auxiliary Doubly Linked List containing all stored key-data pairs.
- Each key-data pair gets two pointers (previous node, next node).
- Table insert & delete modify the Linked List.
- Table traverse uses the Linked List. Result: traverse is $O(n)$.  

![Hash Table data Pointers](image)
Hash Tables
Efficiency — Issues

The Table-remake operation has a similar effect on Hash-Table efficiency to that of reallocate-and-copy on a smart array.

- Constant time becomes amortized constant time.

All reasonable implementations of a Hash Table have **average-case** performance of constant time for retrieve and delete, and also for insert, if no Table-remake is required.

- For the insert operation, this becomes an average case of amortized constant time, if Table-remake operations are done intelligently.

In common Hash Table implementations, **worst-case** performance is linear time for retrieve and delete, and also for insert, if duplicate keys are not allowed.

An important issue is whether a **malicious user** can force worst-case performance.

- A well-chosen hash function makes this difficult.
- The design of such a function is beyond the scope of this class, but information and implementations are not hard to find.
Hash Tables
Efficiency — Comparison

<table>
<thead>
<tr>
<th>Idea #1</th>
<th>Idea #2</th>
<th>Idea #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Priority Queue using Heap</td>
<td>Red-Black Tree</td>
<td>Hash Table: average case</td>
</tr>
<tr>
<td>Retrieve</td>
<td>Constant*</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>Insert</td>
<td>(Amortized)** Logarithmic</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>Delete</td>
<td>Logarithmic*</td>
<td>Logarithmic</td>
</tr>
</tbody>
</table>

*Priority Queue retrieve & delete are not Table operations in their full generality. Only the item with the highest priority can be retrieved/deleted.

**This is logarithmic if (1) the PQ does not manage its own memory, or (2) enough memory is preallocated. Otherwise, occasional linear-time reallocate-and-copy may be required. Time per-operation, averaged over many consecutive operations, will be logarithmic. Thus, “amortized logarithmic”.

***Hash Table insert is constant time in a “double average” sense: when averaged both over all possible inputs and over a large number of consecutive operations.

****This is amortized constant time if both of the following are true: (1) separate chaining is used, and (2) duplicate keys are allowed.
We have another example of average-case vs. worst-case efficiency trade-off.

- One that we saw was Quicksort vs. $O(n \log n)$ sorts. But we do not need to worry about that any more.
- However, Hash Tables vs. balanced search trees is still an issue.

Hash Tables have very good performance for “typical” situations.
- Its occasional drawbacks can be serious.

**When using a Hash Table, do so intelligently.**
Prefix Trees
Background

Consider a list of words.

- In practice, our list might be much longer.
- Alphabetically order the words. Each is likely to have many letters in common with its predecessor.
- That is, consecutive words tend to have a prefix in common.

One easy way to take advantage of this is to store each word as a number followed by letters.

- This method is very suitable for use in a text file that is loaded all at once.
- But it does not support fast look-up by key (word).

A method more suited for in-memory use is a Prefix Tree.

- Also (and, sadly, more commonly) called a “Trie”.
  - For “reTRIEval”.
  - You’re supposed to say “TREE”. 😐
  - I’ve heard “TRY”. 😞
  - Ick.

dig
dog
dot
dote
doting
eggs

0dig
1og
2t
3e
3ing
0eggs

Not a Prefix Tree!
Prefix Trees
Definition [1/2]

A **Prefix Tree** (or **Trie**) is a Table implementation in which the keys are strings.

- We use “string” in a general sense, as in our discussion of Radix Sort.
  - A nonnegative integer is a string of digits.
- The quintessential key type is **words**, as in the previous slide.
- A Prefix Tree is space-efficient when keys tend to share prefixes.

A Prefix Tree is a kind of tree.

- Each node can have one child for each possible character.
- Each node also contains a Boolean value, indicating whether it represents a stored key.
  - Duplicate keys are not allowed.
- Lastly, each node can hold the data associated with a key.
In a Prefix Tree for storing words composed only of lower-case English letters, each node has:

- 26 child pointers (one for each letter).
- A Boolean value
- A spot for the associated data.

The keys in the Prefix Tree to the right are those from our word list:

**dig, dog, dot, dote, doting, eggs.**

- Rather than draw 26 pointers for each node, I have labeled each pointer with the appropriate letter.
- A node with a black circle is one that represents a word in the list.
Prefix Trees
Implementation

How would we implement a Prefix Tree node?

- Example:

```c
struct PTNode {
    (PTNode *) ptrs_[26];  // a .. z ptrs; NULL if none
    bool isWord_;          // true if a word ends here
    DataType data_;        
};
```

- Another possibility:

```c
struct PTNode {
    std::map<char, PTNode *> ptrs_;  // An STL Table implementation
    bool isWord_;                     
    DataType data_;                   
};
```

An RAI class would be good to have here.
See Boost’s `shared_ptr`.
Prefix Trees
Any Good?

Efficiency
- For a Prefix Tree, Table retrieve, insert, and delete all take a number of steps proportional to the length of the key.
- If word length is considered fixed, then all are constant time.
- However, word length is logarithmic in the number of possible words.
  - A hidden logarithm, just like Radix Sort.

A Prefix Tree is a good basis for a Table implementation, when keys are short-ish sequences from a not-too-huge alphabet.
- Words in a dictionary, ZIP codes, etc.
- Just like Radix Sort.

A Prefix Tree is **easy to implement**.
The idea behind Prefix Trees is also used in other data structures.