Notes on Assignment 7
Heaps & Priority Queues in the C++ STL
2-3 Trees

CS 311 Data Structures and Algorithms

Lecture Slides

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Notes on Assignment 7
Introduction

In Assignment 7, you are to write a simple Binary Search Tree (class **`BSTree`**), with copying, retrieve, insert (but not delete!), and the three standard traversals.

Here are some ideas about writing the code.

This is a **node-based structure**.

- The general organization should be much like the Linked List you wrote: a class for the data structure as a whole, and a node class that the client code never sees.
- The node class should have three data members: a data item, and two child pointers: left & right. Each of these pointers is NULL if there is no child.
- Class **`BSTree`** should have two data members: head pointer (pointer to a node) and size.

Notes on Assignment 7
Helper Functions

Some useful private helper functions to write:

`copy`

- Copying is easy to do recursively. But your copy constructor cannot be recursive. So write a private recursive helper function **`copy`**.
- Function **`copy`** takes a pointer to a node, copies the tree rooted at that node, and then returns a pointer to the root of the new tree.
- Do this recursively:
  - If the given pointer is NULL, return NULL (base case). Otherwise, return a new node whose data is the same as the data in the given node, and whose left and right child pointers are gotten by recursive calls to **`copy`** on the given node's left & right child pointers.

`swap`

- As usual: call **`std::swap`** on each data member.

Notes on Assignment 7
Ideas for Writing Functions [1/2]

Copy constructor

- Do everything in initializers. Set the head pointer to the result of calling helper function **`copy`** on the other object's head pointer. Set size to the other object's size.

Copy assignment operator

- Use the swap trick.

`retrieve`

- Call helper function **`find`** with the given key. Return **`true`** if **`find`** returns a non-NULL value. Otherwise, return **`false`**.

`insert`

- Call helper function **`find`** with the given key. Return **`true`** if **`find`** returns a non-NULL value. Otherwise, set the given pointer to point to a new node holding the given key, increment size, and return **`false`**.

Notes on Assignment 7
Ideas for Writing Functions [2/2]

`preorderTraverse`

- Take the iterator **by value** (so that an array decays to a pointer). Then call a private helper function with the given iterator and the head pointer.
- Private helper function: takes iterator **by reference** (since it modifies the iterator) and a pointer to a node. If the pointer is NULL, return. Otherwise, do `*`*iterator*`++ =` *pointer*`->data_`, and then make two recursive calls:
  - One taking *iterator* and *pointer*`->left`.
  - One taking *iterator* and *pointer*`->right`.

`inorderTraverse`, `postorderTraverse`

- Write the same way as function `preorderTraverse`, but do the `*`*iterator*`++ =` *pointer*`->data_` operation in the appropriate place.
- Note that these will require *different* helper functions.

Where Are We? — The Big Problem

Our problem for much of the rest of the semester:

- Store: a collection of data items, all of the same type.
- Operations:
  - Access items [one item: retrieve/find, all items: traverse].
  - Add new item [insert].
  - Eliminate existing item [delete].
- All this needs to be efficient in both time and space.

A solution to this problem is a **container**.

Generic containers: those in which client code can specify the type of data stored.

Review
Binary Search Trees — Efficiency

	B.S.T. (balanced & average case)	Sorted Array	B.S.T. (worst case)
Retrieve	Logarithmic	Logarithmic	Linear
Insert	Logarithmic	Linear	Linear
Delete	Logarithmic	Linear	Linear

Binary Search Trees have poor worst-case performance.

But they have very good performance:

- On average.
- If balanced.
  - But we do not know an efficient way to make them *stay* balanced.

Can we efficiently keep a Binary Search Tree balanced?

- We will look at this question again later.

Unit Overview
Tables & Priority Queues

Major Topics

- ✓ Introduction to Tables ← Lots of lousy implementations
- ✓ Priority Queues
- ✓ Binary Heap algorithms
  - Heaps & Priority Queues in the C++ STL

  } Idea #1: Restricted Table
- 2-3 Trees
- Other balanced search trees

  } Idea #2: Keep a Tree Balanced
- Hash Tables

  } Idea #3: "Magic Functions"
- Prefix Trees
- Tables in various languages

Review
Introduction to Tables [1/2]

What are possible Table implementations?

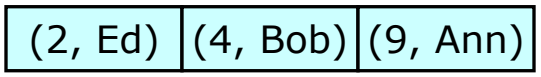
- A Sequence holding key-data pairs.
  - Sorted or unsorted.
  - Array-based or Linked-List-based.
- A Binary Search Tree holding key-data pairs.
  - Implemented using a pointer-based Binary Tree.

Table

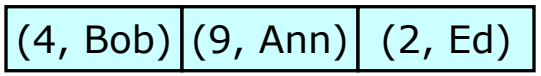
Key	Data
4	Bob
9	Ann
2	Ed

Array Implementations

Sorted

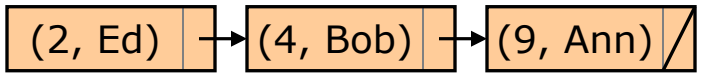


Unsorted



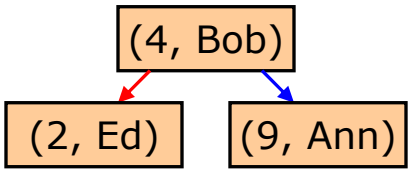
Linked List Implementations

Sorted



Unsorted

Binary Search Tree Implementation



Review
Introduction to Tables [2/2]

	Sorted Array	Unsorted Array	Sorted Linked List	Unsorted Linked List	Binary Search Tree	Balanced (how?) Binary Search Tree
Retrieve	Logarithmic	Linear	Linear	Linear	Linear	Logarithmic
Insert	Linear	Constant???	Linear	Constant	Linear	Logarithmic
Delete	Linear	Linear	Linear	Linear	Linear	Logarithmic

Idea #1: Restricted Table

- Perhaps we can do better if we do not implement a Table in its full generality.

Idea #2: Keep a Tree Balanced

- Balanced Binary Search Trees look good, but how to keep them balanced efficiently?

Idea #3: "Magic Functions"

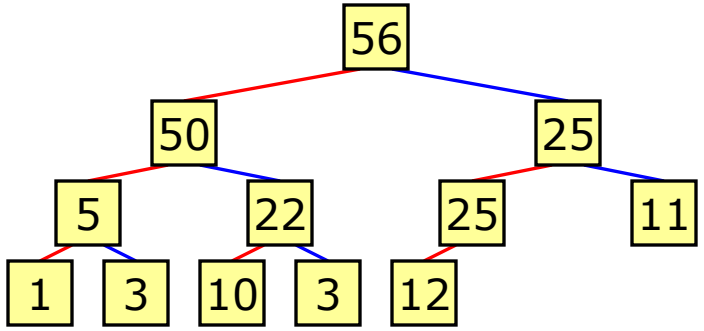
- Use an unsorted array of key-data pairs. Allow array items to be marked as "empty".
- Have a "magic function" that tells the index of an item.
- Retrieve/insert/delete in constant time? (Actually no, but this is still a worthwhile idea.)

We will look at what results from these ideas:

- From idea #1: Priority Queues
- From idea #2: Balanced search trees (2-3 Trees, Red-Black Trees, B-Trees, etc.)
- From idea #3: Hash Tables

Review
Binary Heap Algorithms [1/8]

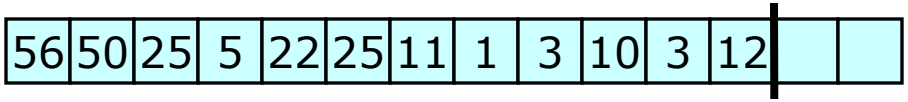
A **Binary Heap** (usually just **Heap**) is a complete Binary Tree in which *no* node has a data item that is less than the data item in either of its children.



Logical Structure

In practice, we often use "Heap" to refer to the array-based complete Binary Tree implementation.

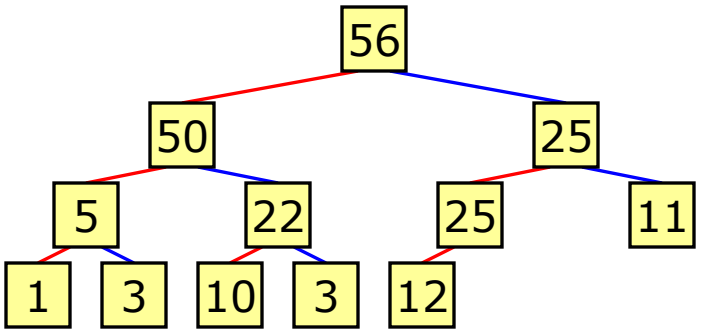
Physical Structure

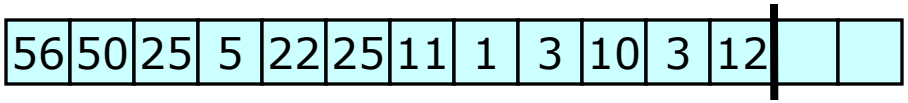


Review
Binary Heap Algorithms [2/8]

We can use a Heap to implement a **Priority Queue**.

- Like a Table, but retrieve/delete only highest key.
  - Retrieve is called "getFront".
- Key is called "priority".
- Insert *any* key-data pair.





Algorithms for the Three Primary Operations

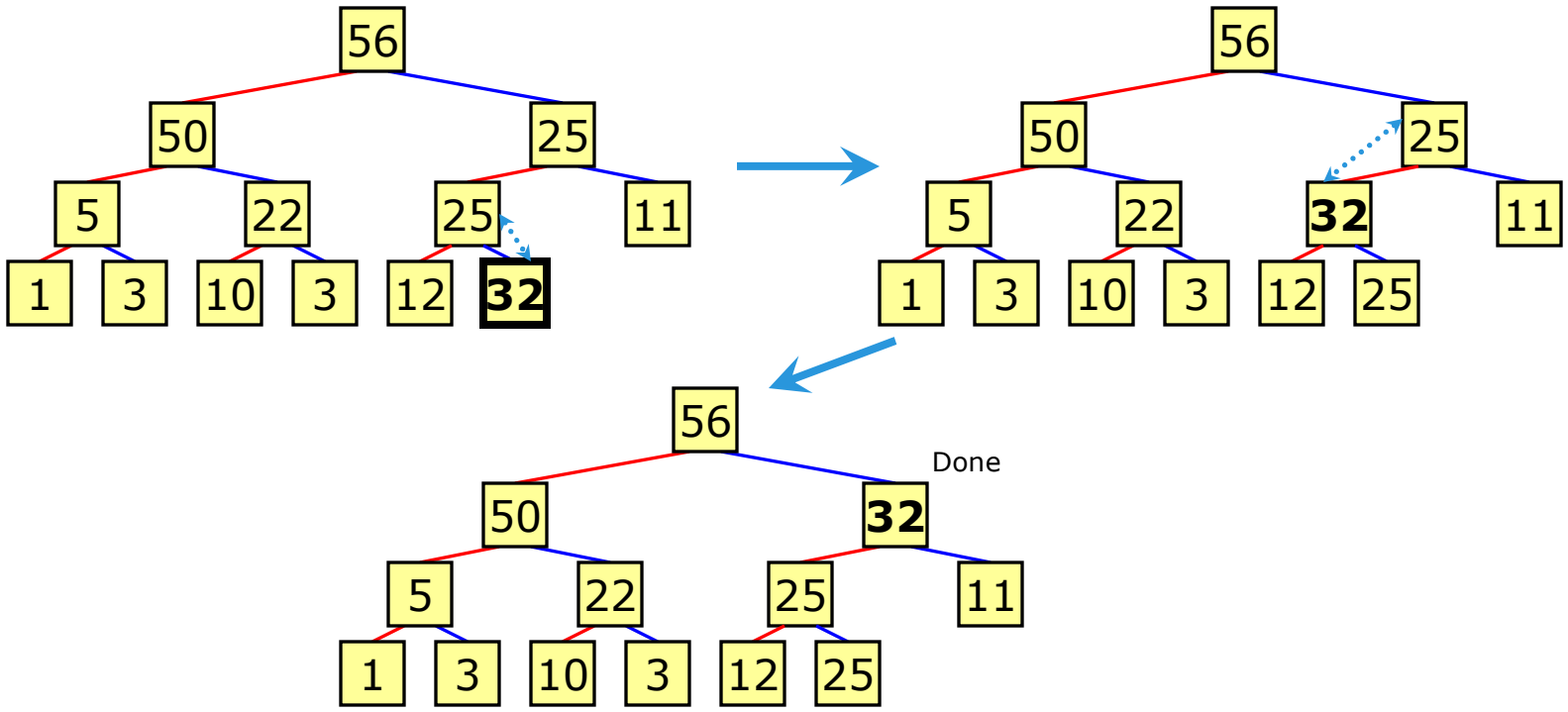
- GetFront
  - Get the root node.
  - Constant time.
- Insert
  - Add new node to end of Heap, "trickle up".
  - Logarithmic time if no reallocate-and-copy required.
    - Linear time if it may be required. Note: Heaps often do *not* manage their own memory, in which case the reallocation will not be part of the Heap operation.
- Delete
  - Swap first & last items, reduce size of Heap, "trickle down" root.
  - Logarithmic time.

Faster than linear time!

Review
Binary Heap Algorithms [3/8]

To insert into a Heap, add new node at the end. Then "trickle up".

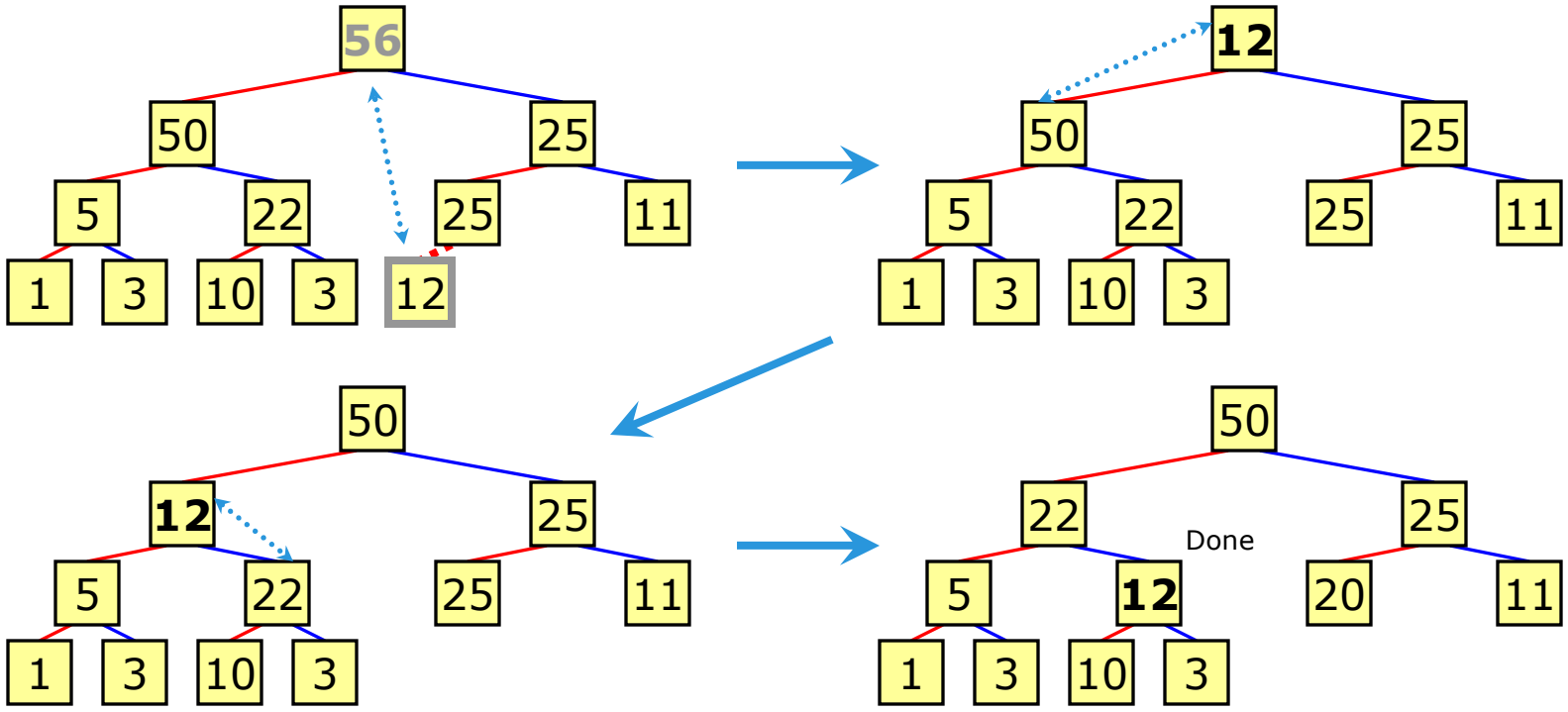
- If new value is greater than its parent, then swap them. Repeat at new position.



Review
Binary Heap Algorithms [4/8]

To delete the root item from a Heap, swap root and last items, and reduce size of Heap by one. Then "trickle down" the new root item.

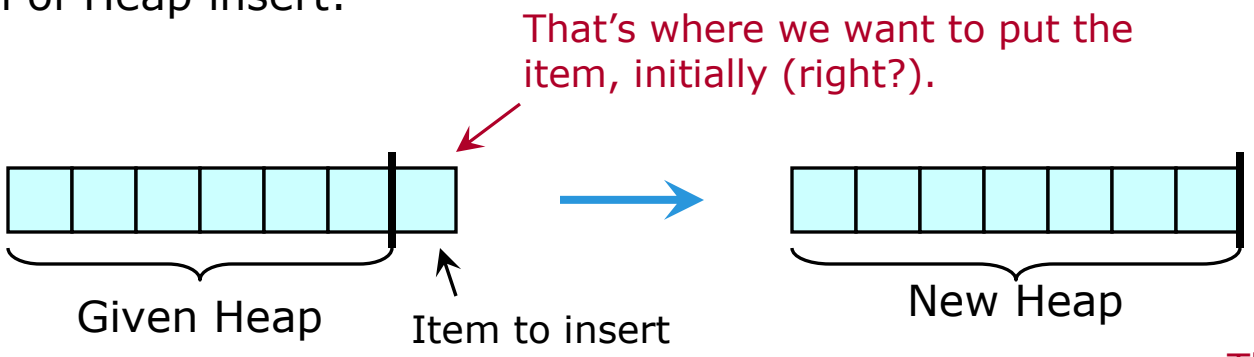
- If the root is ≥ all its children, stop.
- Otherwise, swap the root item with its **largest** child and recursively fix the proper subtree.



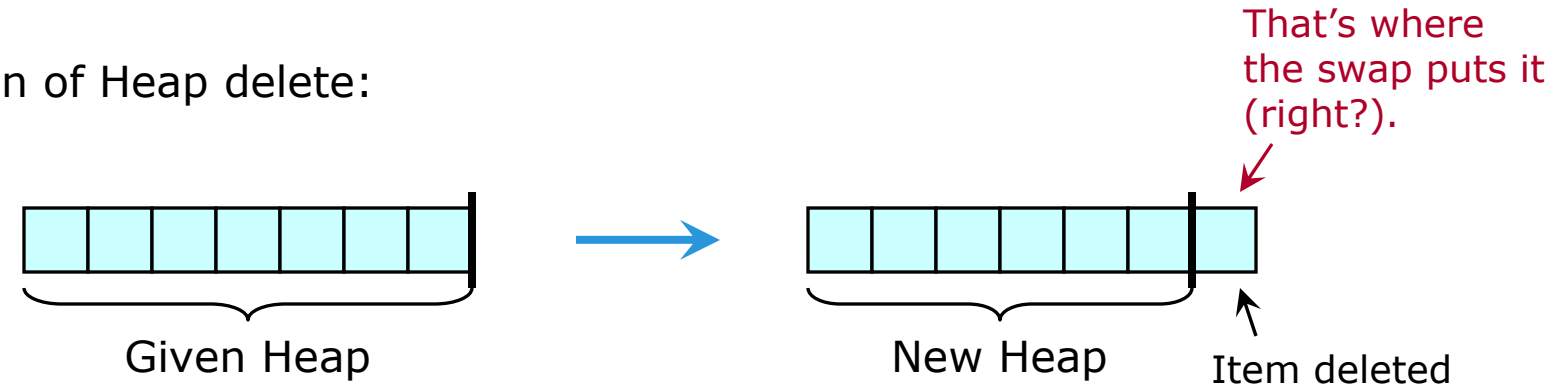
Review
Binary Heap Algorithms [5/8]

Heap insert and delete are usually given a random-access range. The item to insert or delete is last item of the range; the rest is a Heap.

- Action of Heap insert:



- Action of Heap delete:



Note that Heap algorithms can do **all** their work using **swap**.

- This usually allows for both speed and safety.

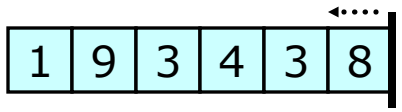
Review
Binary Heap Algorithms [6/8]

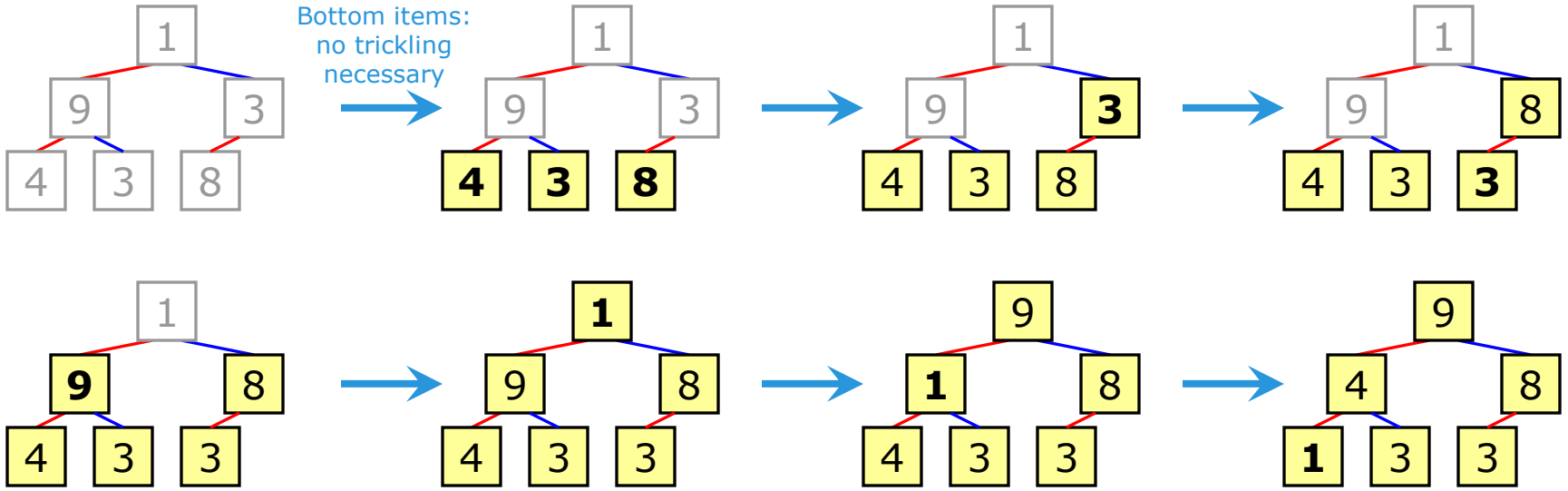
To turn a random-access range (array?) into a Heap, we *could* do $n-1$ Heap inserts.

- Each insert operation is $O(\log n)$, and so making a Heap in this way is $O(n \log n)$.

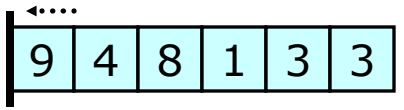
However, we can make a Heap **faster** than this.

- Place each item into a partially-made Heap, in **backwards order**.
- Trickle each item *down* through its descendants.
  - For most items, there are not very many of these.





This Heap "make" method is linear time!



Review
Binary Heap Algorithms [7/8]

Our last sorting algorithm is **Heap Sort**.

- This is a sort that uses Heap algorithms.
- We can think of it as using a Priority Queue, where the priority of an item is its value — except that the algorithm is in-place, using no separate data structure.
- Procedure: Make a Heap, then delete all items, using the delete procedure that places the deleted item in the top spot.
- We do a **make** operation, which is $O(n)$, and n getFront/delete operations, each of which is $O(\log n)$.
- Total: $O(n \log n)$.

Review
Binary Heap Algorithms [8/8]

Efficiency ☺

- Heap Sort is $O(n \log n)$.

Requirements on Data ☹

- Heap Sort requires random-access data.

Space Usage ☺

- Heap Sort is in-place.

Stability ☹

- Heap Sort is not stable.

Performance on Nearly Sorted Data 😐

- Heap Sort is not significantly faster or slower for nearly sorted data.

We have seen these together before (Iterative Merge Sort on a Linked List), but never for an array.

Notes

- Heap Sort can be generalized to handle sequences that are modified (in certain ways) in the middle of sorting.
- Recall that Heap Sort is used by Introsort, when the depth of the Quicksort recursion exceeds the maximum allowed.

Heaps & Priority Queues in the C++ STL
Heap Algorithms

The C++ STL includes several Heap algorithms.

- These operate on ranges specified by pairs of random-access iterators.
  - **Any** random-access range can be a Heap: array, vector, deque, part of these, etc.
- An STL Heap is a Maxheap with an optional client-specified comparison.
- Heap algorithms are used by STL Priority Queues (**`std::priority_queue`**).

Example: **`std::push_heap`** (in **`<algorithm>`**) inserts into an existing Heap.

- Called as **`std::push_heap(`***first*`, `*last*`)`**.
- Assumes [*first*, *last*) is nonempty, and [*first*, *last*`-1`) is already a Heap.
- Inserts **`*(`***last*`-1)`** into the Heap.

Similarly:

- **`std::pop_heap`**
  - Heap delete operation. Puts the deleted element in **`*(`***last*`-1)`**.
- **`std::make_heap`**
  - Make a range into a Heap.
- **`std::sort_heap`**
  - Is given a Heap. Does a bunch of **`pop_heap`** calls.
  - Calling **`make_heap`** and then **`sort_heap`** does Heap Sort.
- **`std::is_heap`**
  - Tests whether a range is a Heap.

Heaps & Priority Queues in the C++ STL
`std::priority_queue` — Introduction

The STL has a Priority Queue: **`std::priority_queue`**, in **`<queue>`**.

- Once again, STL documentation calls **`std::priority_queue`** a "container adapter", not a "container".

As with **`std::stack`** and **`std::queue`**, **`std::priority_queue`** is a wrapper around a container that you choose.

```
std::priority_queue<T, container<T> >
```

- "**`T`**" is the value type.
- "*container*`<T>`" can be any standard-conforming **random-access** sequence container with value type **`T`**.
- In particular "*container*" can be **`std::vector`**, **`std::deque`**, or **`std::basic_string`**.
  - But not **`std::list`**.

container defaults to **`std::vector`**.

```
std::priority_queue<T>
    // = std::priority_queue<T, std::vector<T> >
```

Heaps & Priority Queues in the C++ STL
`std::priority_queue` — Members

The member function names used by **`std::priority_queue`** are the same as those used by **`std::stack`**.

- Not those used by **`std::queue`**.
- Thus, **`std::priority_queue`** has "**`top`**", not "**`front`**".

Given a variable **`pq`** of type **`std::priority_queue<T>`**, you can do:

- **`pq.top()`**
- **`pq.push(`***item*`)`**
  - "*item*" is some value of type **`T`**.
- **`pq.pop()`**
- **`pq.empty()`**
- **`pq.size()`**

Heaps & Priority Queues in the C++ STL
`std::priority_queue` — Comparison

How do we specify an item's priority?

- We really don't need to know an item's priority; we only need to know, given two items, which has the **higher** priority.
- Thus, we use a comparison, which defaults to **`operator<`**.
- A third, optional template parameter is a "comparison object":

```
std::priority_queue<T, std::vector<T>,
                    compare>
```

- Comparison objects work the same as those passed to STL sorting algorithms (**`std::sort`**, etc.) and STL Heap algorithms.
- So, for example, a priority queue of **`int`**s whose highest priority items are those with the lowest value, would have the following type:

```
std::priority_queue<int, std::vector<int>,
                    std::greater<int>()>
```

Overview of Advanced Table Implementations

We will cover the following advanced Table implementations.

- Balanced Search Trees
  - Binary Search Trees are hard to keep balanced, so to make things easier we allow more than 2 children:
    - **2-3 Tree**
      - Up to 3 children
    - **2-3-4 Tree**
      - Up to 4 children
    - **Red-Black Tree**
      - Binary-tree representation of a 2-3-4 tree
  - Or back up and try a balanced Binary Tree again:
    - **AVL Tree**
- Alternatively, forget about trees entirely:
  - **Hash Tables**
- Finally, "the Radix Sort of Table implementations":
  - **Prefix Tree**

2-3 Trees
Introduction & Definition [1/3]

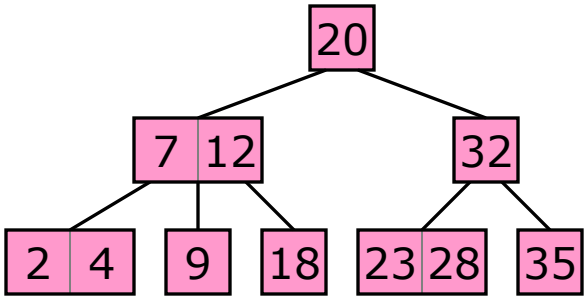
Obviously (?) a Binary Search Tree is a useful idea. The problem is keeping it balanced.

- Or at least keeping the height small.

It turns out that small height is much easier to maintain if we allow a node to have more than 2 children.

But if we do this, how do we maintain the "search tree" concept?

- We generalize the idea of an inorder traversal.
- For each pair of consecutive subtrees, a node has one data item lying between the values in these subtrees.



2-3 Trees
Introduction & Definition [2/3]

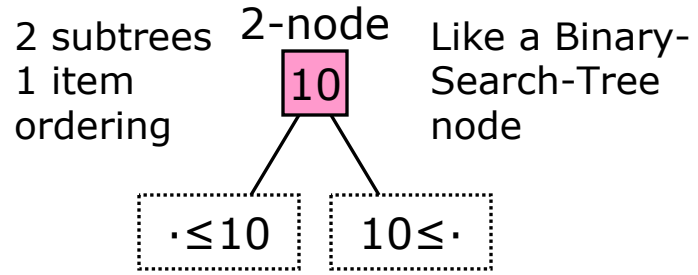
A Binary-Search-Tree style node is a **2-node**.

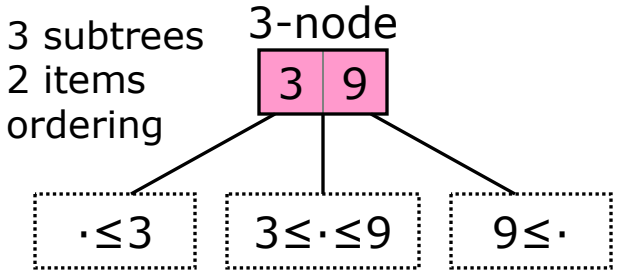
- This is a node with 2 subtrees and 1 data item.
- The item's value lies between the values in the two subtrees.

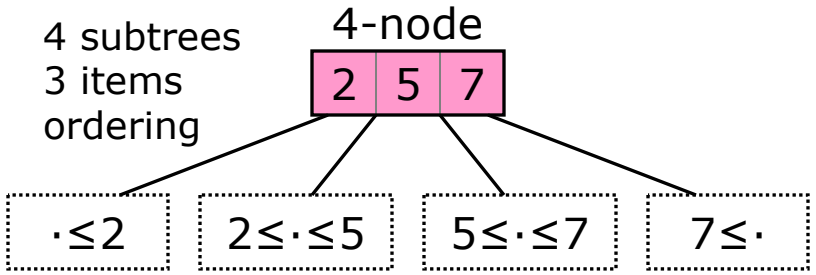
In a "2-3 Tree" we also allow a node to be a **3-node**.

- This is a node with 3 subtrees and 2 data items.
- Each of the 2 data items has a value that lies between the values in the corresponding pair of consecutive subtrees.

Later, we will look at "2-3-4 trees", which can also have **4-nodes**.



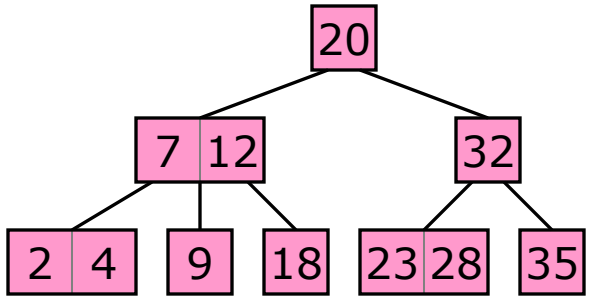




2-3 Trees
Introduction & Definition [3/3]

A **2-3 Search Tree** (generally we just say **2-3 Tree**) is a tree with the following properties.

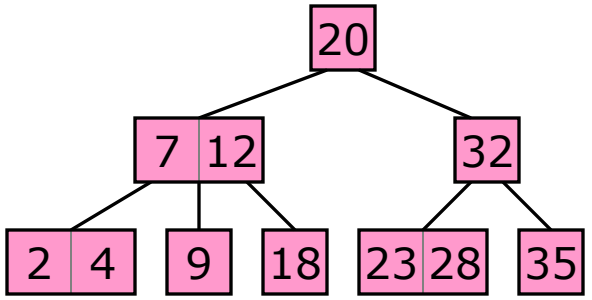
- All nodes contain either 1 or 2 data items.
  - If 2 data items, then the first is ≤ the second.
- All leaves are at the same level.
- All non-leaves are either *2-nodes* or *3-nodes*.
  - They must have the associated order properties.



2-3 Trees
Operations — Traverse & Retrieve

How do we **traverse** a 2-3 Tree?

- We generalize the procedure for doing an **inorder traversal** of a Binary Search Tree.
  - For each leaf, go through the items in it.
  - For each non-leaf 2-node:
    - Traverse subtree 1.
    - Do item.
    - Traverse subtree 2.
  - For each non-leaf 3-node:
    - Traverse subtree 1.
    - Do item 1.
    - Traverse subtree 2.
    - Do item 2.
    - Traverse subtree 3.



- This procedure lists all the items in sorted order.

How do we **retrieve** by key in a 2-3 Tree?

- Start at the root and proceed downward, making comparisons, just as in a Binary Search Tree.
- 3-nodes make the procedure *slightly* more complex.

2-3 Trees
Operations — Insert & Delete

How do we **insert** & **delete** in a 2-3 Tree?

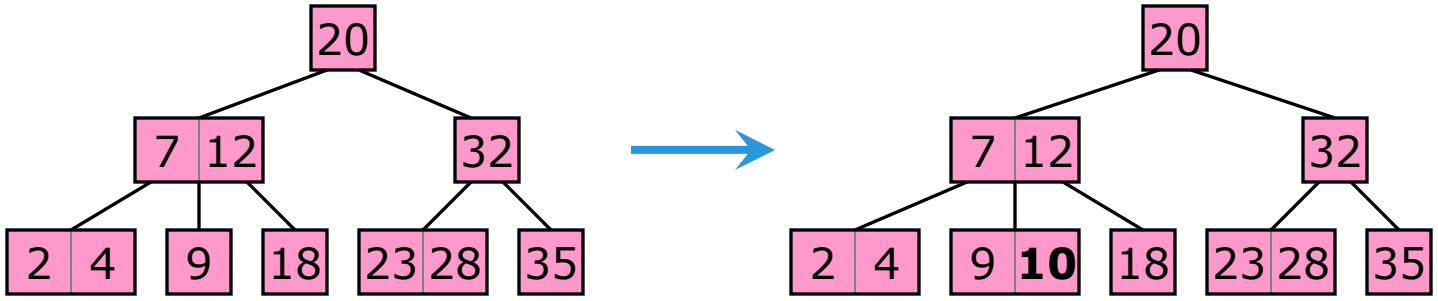
- These are the tough problems.
- It turns out that both have efficient [$O(\log n)$] algorithms, which is why we like 2-3 Trees.

2-3 Trees
Operations — Insert [1/4]

Ideas in the 2-3 Tree **insert** algorithm:

- Start by adding the item to the appropriate leaf.
- Allow nodes to expand when legal.
- If a node gets too big (3 items), split the subtree rooted at that node and propagate the **middle** item upward.
- If we end up splitting the entire tree, then we create a new root node, and all the leaves advance one level simultaneously.

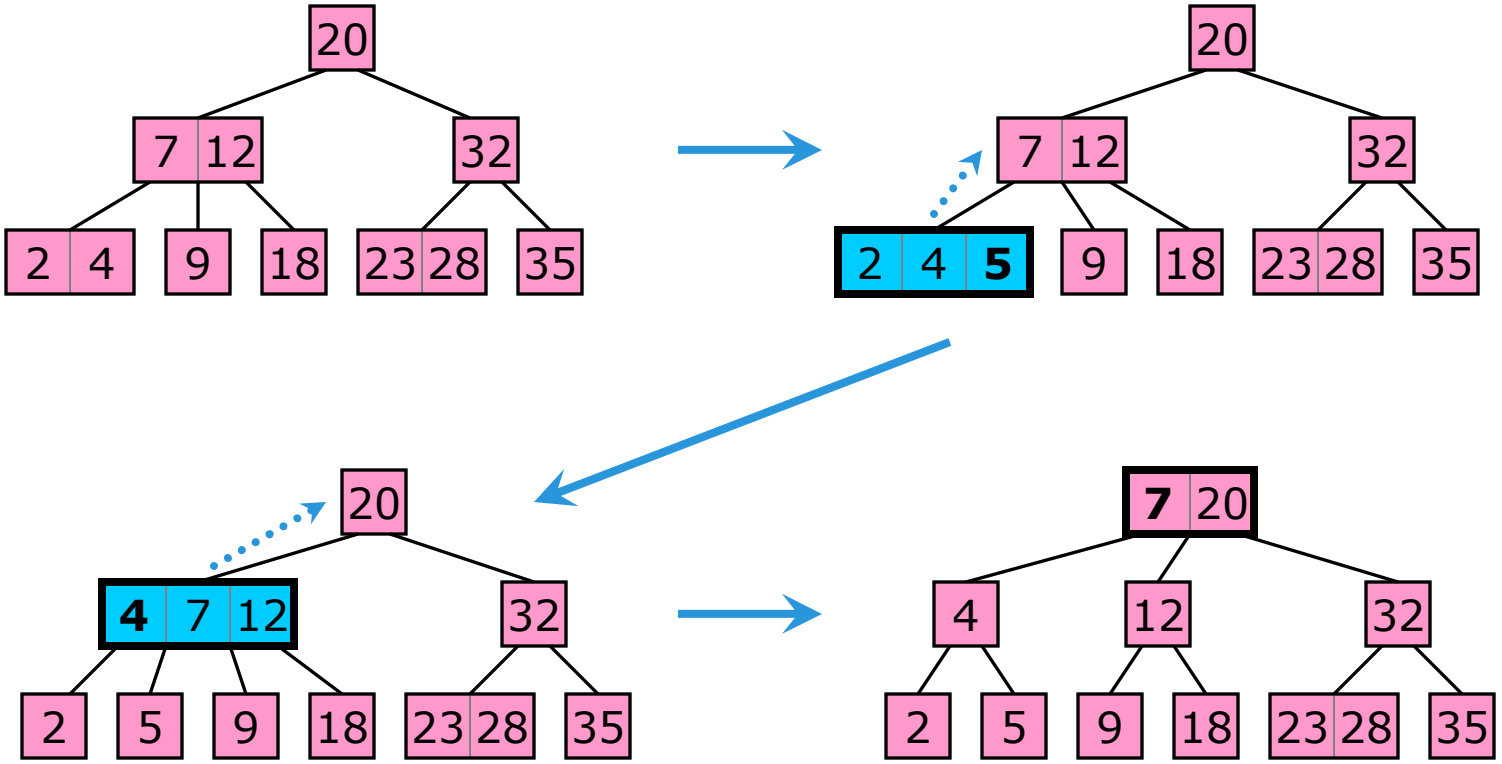
Example 1: Insert 10.



Operations — Insert [2/4]

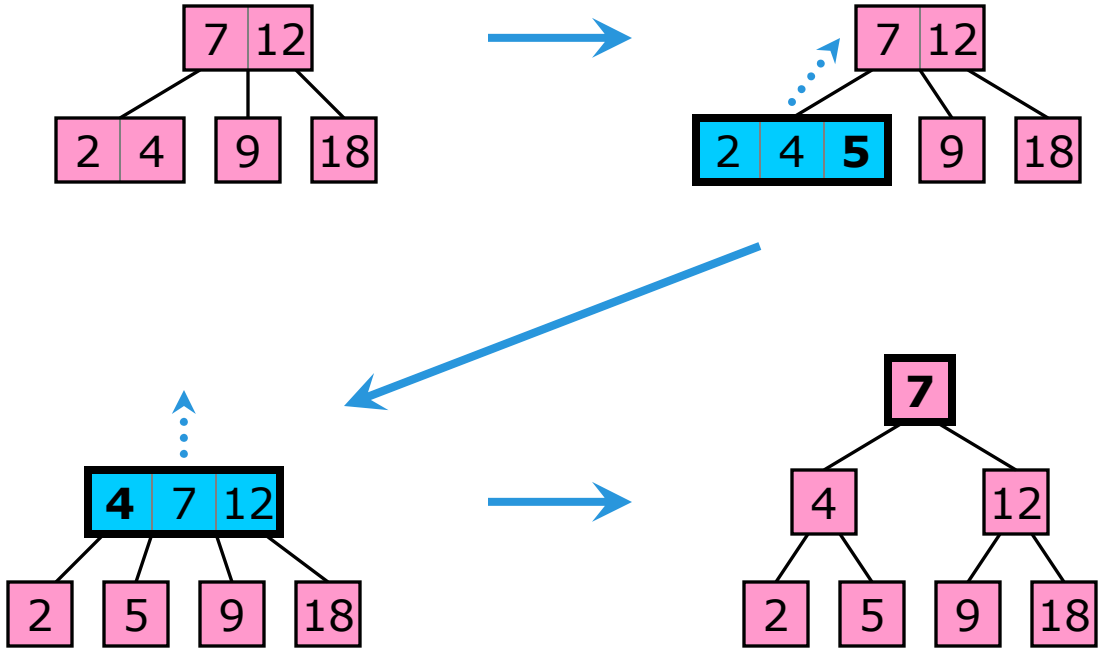
Example 2: Insert 5.

- Over-full nodes are blue.



Example 3: Insert 5.

- Here we see how a 2-3 Tree increases in height.



2-3 Tree **Insert** Algorithm (outline)

- Find the leaf the new item goes in.
  - Note: In the process of finding this leaf, you may determine that the given key is already in the tree. If you do, act accordingly.
- Add the item to the proper node.
- If the node is overfull, then split it (dragging subtrees along, if necessary), and move the middle item up:
  - If there is no parent, then make a new root. Done.
  - Otherwise, add the moved-up item to the parent node. To add the item to the parent, do a recursive call to the insertion procedure.

Deleting from a 2-3 Tree is similar to inserting.

- We will use the recursive-thinking idea to avoid describing every detail of the process.
- We try to delete from a leaf. If it does not work, rearrange.
- If that does not work, bring an item from the parent down. This is deleting from the parent. Recurse (or reduce the height and we are done).
- As with inserting, we start at a leaf and work our way up.

2-3 Trees
Operations — Delete [2/8]

Observation

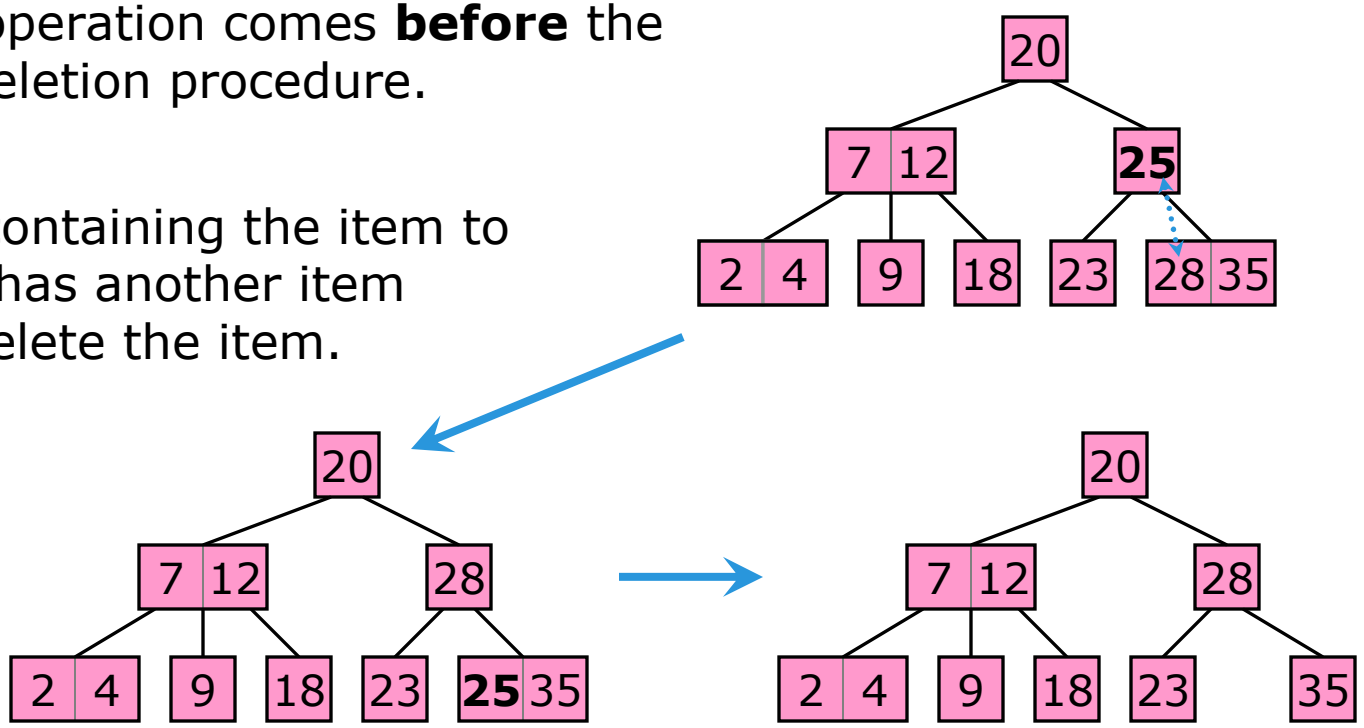
- We can always start our deletion at a leaf.
- If the item to be deleted is not in a leaf, swap it with its "inorder" successor.
  - It must have one. (Why?)
- This swap operation comes **before** the recursive deletion procedure.

Easy Case

- If the leaf containing the item to be deleted has another item in it, just delete the item.

Example

- Delete 25.

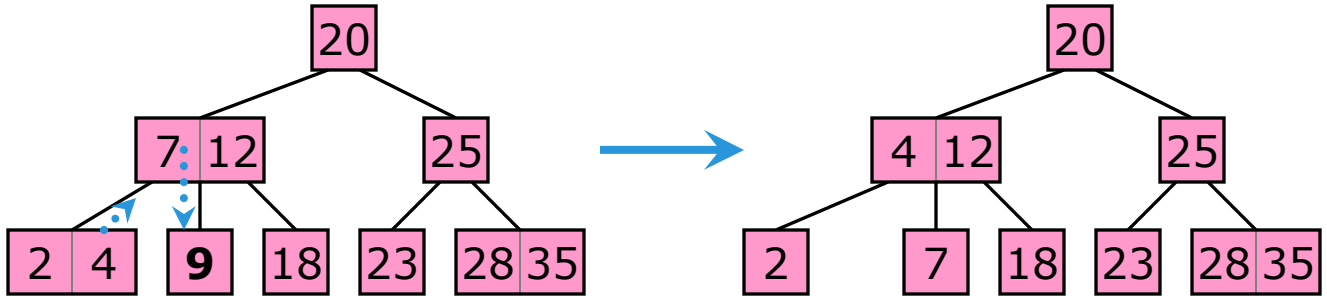


Operations — Delete [3/8]

Semi-Easy Case

- Suppose the item to be deleted is in a node that contains no other item.
- If, next to this node, there is a sibling that contains 2 items, we can rearrange using the parent.

Example: Delete 9.

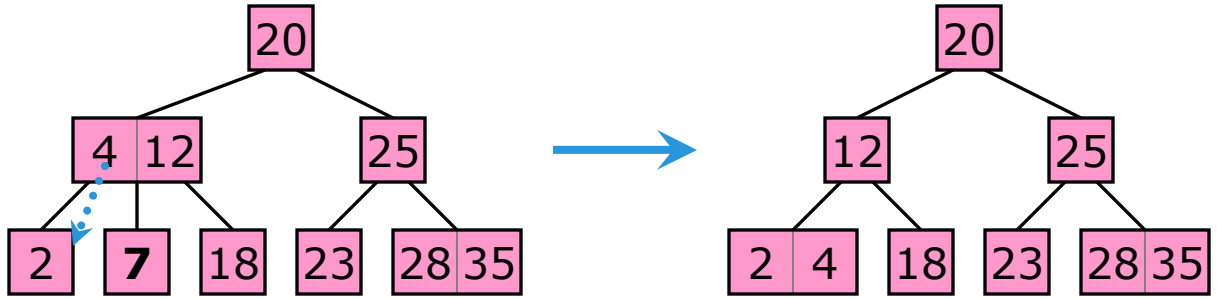


Operations — Delete [4/8]

Hard Case

- If the item to be deleted is in a node with no other item, and there are no nearby 2-item siblings, then we must bring down an item from the parent and place it in a nearby sibling node.
- We need to join nodes/subtrees to make the invariants work.

Example: Delete 7.



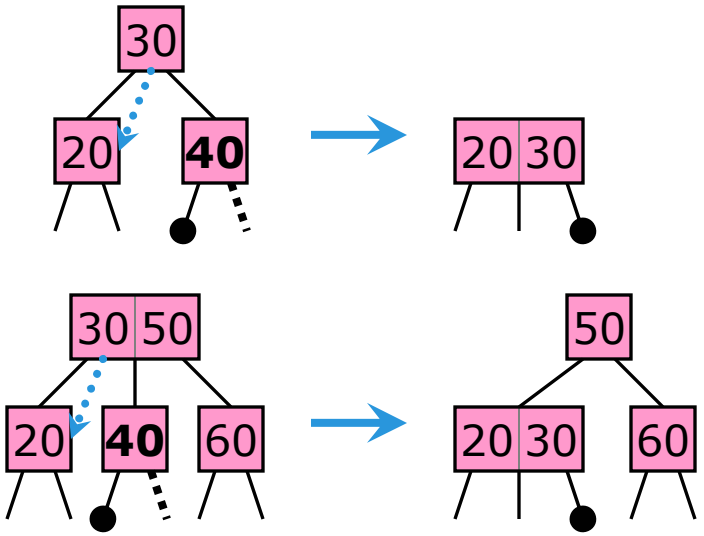
In the above example, recursively "delete" 4 from the tree consisting of the first two levels. Since 4's node has another item in it, this is the easy case; we simply get rid of 4 (and then put it in the node containing 2).

2-3 Trees
Operations — Delete [5/8]

If we do a recursive delete above the leaf level, where do "orphaned" subtrees go?

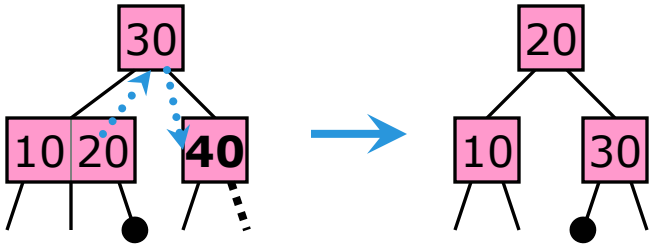
Consider two Hard Case examples.

- We delete 40. Why? Because one of its subtrees is going away. What do we do with the other subtree?
- Answer: Make it a subtree of the item we bring down.



Consider a Semi-Easy Case example.

- Again, we delete 40. One of its subtrees is going away. 30 is coming down to replace it. 20 is going up. What do we do with the right-subtree of 20?
- Answer: Make it the left subtree of 30.



Idea: There is always exactly one spot available for an orphaned subtree. Put it in that spot.

2-3 Tree **Delete** Algorithm (outline)

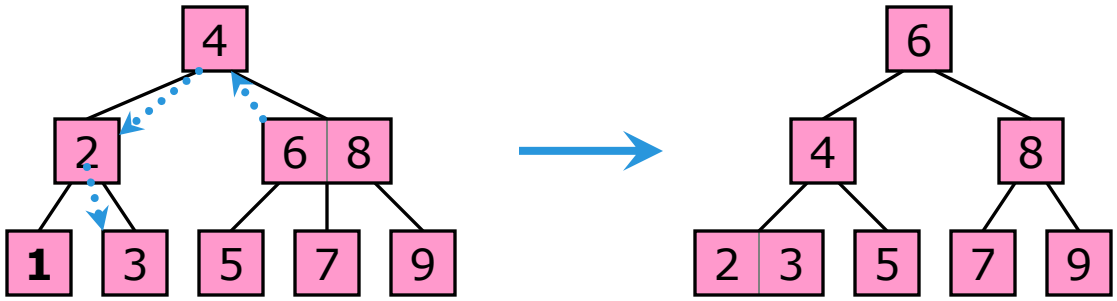
- Find the node holding the given key.
  - Note: In the process of this search, you may determine that the given key is not in the tree. If you do, act accordingly.
- If the above node is not a leaf, then swap its item with its successor in the traversal ordering. Continue with the deletion procedure: delete the given key from its new (leaf) node.
- 3 Cases
  - **Easy Case** (item shares a node with another item). Delete item. Done.
  - **Semi-Easy Case** (otherwise: item has a consecutive sibling holding 2 items). Do rotation: sibling item up, parent down, to replace the item to be deleted. Done.
  - **Hard Case** (otherwise). Eliminate the node holding the item, and move item from the parent down, adding it to consecutive sibling node. Eliminate item from parent using a recursive call to the deletion procedure (dragging subtrees along).

Operations — Delete [7/8]

A few more examples.

Example: Delete 1.

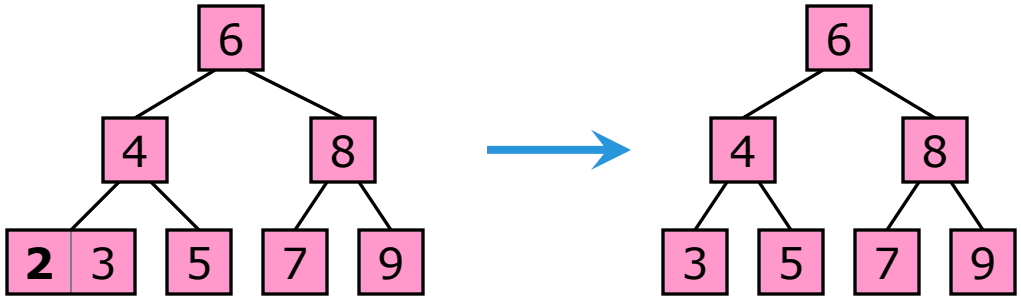
- 1 is "Hard Case", so we bring down the parent (recursively "delete" 2) and join it with 3 in a single node.
- 2 is "Semi-Easy Case", so rotate (6 to 4 to 2).
- The 5 is orphaned. We make it the right child of 4.



Operations — Delete [8/8]

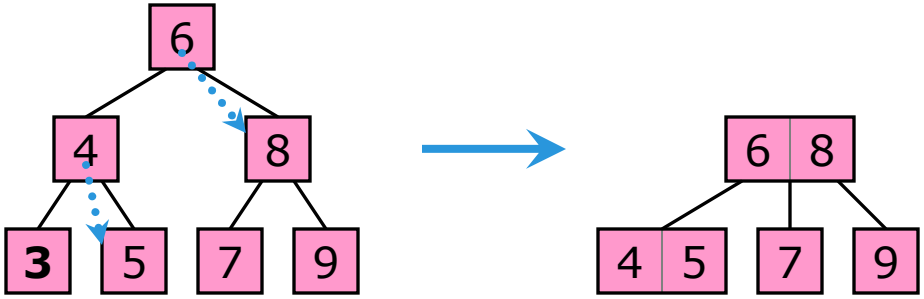
Example: Delete 2.

- This is "Easy Case".



Example: Delete 3.

- This is "Hard Case". We need to bring down 4 and join it with 5.
- 4 is "Hard Case". We need to bring down 6 and join it with 8.
- 6 is the root. We reduce the height of the tree.



2-3 Trees
TO BE CONTINUED …

2-3 Trees will be continued next time.