1. Let γ be a unit speed curve into a surface M with tangent vector T. Suppose the surface is orientable, and has unit normal U. At any point on the curve where $T \equiv \dot{\gamma} \neq 0$ we can decompose

$$\ddot{\gamma} = aT + b(U \times T) + cU$$

for unique constants a,b,c. Moreover, $a = 0$ since $||\dot{\gamma}||^2$ is constant. Recall that the curvature κ of γ in $\mathbb{R}^3$ is

$$\kappa = ||\ddot{\gamma}|| = \sqrt{b^2 + c^2}.$$

Thus we can write

$$\ddot{\gamma} = \kappa \cos(\phi)(U \times T) + \kappa \sin(\phi) U$$

for some angle ϕ, uniquely defined up to multiples of 2π. We called the quantity

$$\kappa_g = \kappa \cos(\phi)$$

the geodesic curvature of γ and

$$\kappa_n = \kappa \sin(\phi)$$

the normal curvature of γ. The point of this exercise is to generalize the notion of geodesic curvature a little and connect it to the notion of covariant derivatives. You should note that geodesic curvature, as defined, is a signed quantity, like planar curvature. We can do this because the surface is orientable, and hence we can make a global choice of a unit tangent vector perpendicular to T.

a) Suppose γ is a not-necessarily unit speed curve in M. We define the geodesic and normal curvatures of γ to be the corresponding curvatures of a unit speed reparameterization of γ. Show that normal and geodesic curvatures of γ can be computed via

$$k_n = \frac{\langle \gamma'', U \rangle}{\langle \gamma', \gamma' \rangle}$$
$$k_g = \frac{\langle \gamma'', U \times \gamma' \rangle}{[\langle \gamma', \gamma' \rangle]^{3/2}}.$$

Hint: Recall that a unit speed reparameterization β of γ satisfies $\beta(s(t)) = \gamma(t)$ where s is the arclength function of γ.

b) Let γ be a regular but not-necessarily unit speed curve in M. Show that

$$\gamma'' = \mu' T + k_g \mu^2 (U \times T) + k_n \mu^2 U$$

where T is the unit tangent to γ and $\mu = |\gamma'|$.

c) Recall that we defined that a curve γ is a geodesic if its acceleration is always in the normal direction. Show that this implies γ is a geodesic if and only if γ' is parallel along γ.

d) Demonstrate that

$$\nabla^M_\gamma \gamma' = \mu' T + k_g \mu^2 (U \times T)$$

and that γ is a geodesic if and only if γ is constant speed and κ_g is zero along γ.

2. Show that if v and w are vector fields defined along γ that $(v \cdot w)' = v \cdot \nabla^M_\gamma w + w \cdot \nabla^M_\gamma v$.

3. Suppose γ is a regular curve in the orientable surface M with normal U. Suppose v is parallel transported along γ. Show that $v^\perp = U \times v$ is also parallel transported along γ, and that $\{v, v^\perp\}$ is an orthonormal basis along γ.

4. With the same notation as in the previous problem, at each point on the curve we can write $\dot{\gamma} = \mu(\cos(\theta)v + \sin(\theta)v^\perp)$, where μ is the speed of the curve and θ is a function uniquely defined up to multiples of 2π. Show that $\kappa_g = \mu^2\dot{\theta}$. In particular, for unit speed curves, $\kappa_g = \dot{\theta}$.

5.

a) Show that for all $n \in \mathbb{Z}$, $\lim_{x\to 0^+} x^n e^{-1/x^2} = 0$.

b) Define $\phi(x) = e^{-1/x^2}$ for $x > 0$ and $\phi(x) = 0$ for $x \le 0$. Show that ϕ is differentiable to all orders at 0 and that all its derivatives vanish there. Hence ϕ is a smooth function.

c) Construct a smooth function that is even, non-negative, equals 1 at 0, and vanishes outside of $(-1, 1)$.

d) Construct a smooth function that is non-negative, has values in $[0, 1]$, is equal to zero on $(-\infty, 0)$ and is equal to 1 on $(1, \infty)$.

6. Suppose γ is a curve from p to q and $\hat{\gamma}$ is a reparameterization of γ. Show that $\Pi^{\hat{\gamma}}_{pq} = \Pi^{\gamma}_{pq}$

7. Compute the inverse of Π^{γ}_{pq} by explicitly constructing a curve β with $\Pi^{\beta}_{pq} = (\Pi^{\gamma}_{pq})^{-1}$