In class we showed that the area between the x-axis, the graph of $y = x^2$ and the lines $x = 0$ and $x = 1$ can be approximated by

$$R_N = \sum_{k=1}^{N} \frac{k^2}{N^3} = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$$

and that the area is given by

$$\lim_{N\to\infty} R_N = \frac{1}{3}.$$

In this worksheet we will find a similar formulas for other regions. In all cases the steps are the same. We break the interval $[a, b]$ up into N subintervals, and compute the sum of areas of rectangles over the subintervals.

- Determine the size Δx of each subinterval.
- Determine formulas for the endpoints x_k in terms of a, k, and Δx.
- Determine formulas for the sample points c_k in each subinterval. Your choice of c_k will depend on your choice of rule.
- Determine the height h_k of the k^{th} rectangle.
- Determine the area A_k of the k^{th} rectangle.
- Write the sum of the areas down using sigma notation.
- Use formulas and other summation trickery to compute the sum exactly.
- Take a limit as $N \to \infty$ to compute the area.

Here are the regions

Region	$f(x)$	a	b	Rule
I	x^2	0	1	Right-hand
II	x^2	0	1	Left-hand
III	x	1	4	Right-hand
IV	x	1	4	Mid-point
V	$x - x^2$	0	1	Right-hand

We did region I in class together, and I've filled in the the table below with the results of our computations. Fill in the table similarly for the other four regions.

Region	I	II	III	IV	V
Δx	$\frac{1}{N}$				
x_k	$\frac{1}{N}$				
c_k	$\frac{1}{N}$				
h_k	$\left(\frac{k}{N}\right)^2$				
A_k	$\frac{k^2}{N^3}$				
Sigma notation	$\sum_{k=1}^{N} \frac{k^2}{N^3}$				
Simplified	$\frac{1}{3}+\frac{1}{2N}+\frac{1}{6N^2}$				
Area	$\frac{1}{3}$				

Challenge: compute

$$\lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1-\left(\frac{k}{N}\right)^2}.$$